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Liftings and Weak Liftings of Modules

MAURICE AUSLANDER, SONGQING DING, AND ØYVIND SOLBERG

Department of Mathematics, Brandeis University, Waltham, Massachusetts 02254

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INTRODUCTION

Throughout this paper we assume that all modules are finitely generated (left) modules over a noetherian R-algebra Λ for a commutative local ring R. By a noetherian R-algebra Λ, we mean that Λ is finitely generated as an R-module and that R is a noetherian ring. We denote by mod Λ the category of all finitely generated Λ-modules.

This paper is devoted to studying liftings and weak liftings of modules, which are defined in the following way. Let $\Lambda \to \Gamma$ be a homomorphism of rings and let M be in mod Γ. Then L in mod Λ is called a *lifting* of M to Λ if (a) $M \simeq \Gamma \otimes_\Lambda L$ and (b) $\operatorname{Tor}_i^\Lambda(\Gamma, L) = (0)$ for all $i \geqslant 1$. The Γ-module M is said to be *liftable* to Λ, when such a Λ-module L exists. If M is only a direct summand of $\Gamma \otimes_\Lambda L$ for some Λ-module L satisfying (b), then L is a *weak lifting of M* to Λ and M is said to be *weakly liftable* to Λ. The notion of a lifting of a module has existed for some time, but the notion of a weak lifting of a module is introduced in this paper. A notion of a lifting has been studied in modular representation theory. Let $\Lambda = RG$ where R is a complete discrete valuation ring with maximal ideal generated by a prime number p and G is a finite group such that p divides the order of G. Assume that the characteristic of R is zero and that the characteristic of the residue class field $R/(p)$ is p. Then a lifting of a module M in mod $\Lambda/p\Lambda$ is a Λ-lattice L such that $M \simeq L/pL$. It is not hard to see that this notion of a lifting is a special case of the general definition given above and many of the results in this paper are in fact inspired by the work of J. A. Green in [9]. An important part of this paper is to introduce and study the properties of weak liftings. The notion of a weak lifting of a module not only is a natural generalization of the notion of a lifting of a module, but we show that notion of weak liftability of a module corresponds to that the module has an infinitesimal deformation, a notion occurring in deformation theory.

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Let Λ be a noetherina R-algebra where R is a commutative complete local ring with maximal ideal $\underline{m}$. Let $\{x_1, x_2, ..., x_t\}$ be a Λ-regular sequence in $\underline{m}$ and $\Gamma = \Lambda/(x_1, x_2, ..., x_t)\Lambda$. The main aim is to prove that if $\operatorname{Ext}^2_\Gamma(M, M) = (0)$ for M in mod Γ, then M is liftable to Λ. This is done by first considering the lifting problem in the situations $\Lambda \to \Lambda/(x) = \Lambda_1$ and $\Lambda_i = \Lambda/(x^i) \to \Lambda_1$, where x is a Λ-regular element in $\underline{m}$. In this situation we characterize Λ_1-modules liftable to Λ in terms of a lifting sequence and obtain homological obstructions for lifting a module. In particular, we show what seems to be folklore to deformation theorists, namely, that if $\operatorname{Ext}^2_{\Lambda_1}(M, M) = (0)$, then M is liftable to Λ (see [11]). We end this section by giving some applications of our main result to module theory. For these applications assume that Λ has finite global dimension. Then we show that if $\operatorname{Ext}^2_\Gamma(M, M) = (0)$ for a Γ-module M, then $pd_\Gamma M < \infty$. Also, if $\operatorname{Ext}^i_\Gamma(M \oplus \Gamma, M \oplus \Gamma) = (0)$ for all $i \geqslant 1$ and M in mod Γ, then M is projective. Examples of such rings Γ where these results apply are complete intersections and therefore also group rings of finite abelian groups. Since we consider liftings in the situations $\Lambda \to \Lambda_1$, $\Lambda_i \to \Lambda_1$, and $\Lambda \to \Lambda/(x_1, x_2, ..., x_t)\Lambda$, this justifies the generality in the definition of a lifting of a module.

The setting in Section 2 is the same as in Section 1. Here, the main object is to study realizations of liftings of a Λ_1-module M and when it is unique up to isomorphism. In particular, we show that every lifting of M to Λ is a factor module of $\Omega_\Lambda(M)$, where $\Omega_\Lambda(M)$ denotes the first syzygy of M over Λ given by the projective cover of M over Λ. Further, we prove the following result which seems to be folklore to deformation theorists. If $\operatorname{Ext}^1_{\Lambda_1}(M, M) = (0)$ for a Λ_1-module M liftable to Λ, then the lifting to Λ of M is unique up to isomorphism (see [11]). This result is also generalized to the situation $\Lambda \to \Gamma$, where $\Gamma = \Lambda/(x_1, x_2, ..., x_t)\Lambda$ for a Λ-regular sequence $\{x_1, x_2, ..., x_t\}$ in $\underline{m}$. Finally, we show that every Λ_1-module liftable to Λ has a unique lifting to Λ up to isomorphism if and only if $x \cdot \operatorname{Ext}^1_\Lambda(L, L') = (0)$ for all Λ-modules L and L' for which x is regular and every weakly Λ-liftable Λ_2-lifting of a Λ_1-module liftable to Λ is unique up to isomorphism.

In Section 3 we introduce the notion of a weak lifting of a module. Here we assume that Λ is a noetherian R-algebra over a commutative local ring R (not necessarily complete) with maximal ideal $\underline{m}$. Let $\{x_1, x_2, ..., x_t\}$ be a Λ-regular sequence of central elements in Λ and denote by I the ideal $(x_1, x_2, ..., x_t)\Lambda$ in Λ. Let $\Gamma = \Lambda/I\Lambda$ and $\Gamma_2 = \Lambda/I^2\Lambda$. Then the aim is to characterize Γ-modules weakly liftable to Λ. In fact, we show that the following are equivalent for a Γ-module M, (a) the module M is weakly liftable to Λ, (b) the module M is liftable to Γ_2, and (c) the module M is isomorphic to a direct summand of $\Omega'_\Lambda(M)/I\Omega'_\Lambda(M) \oplus Q/IQ$ for some projective Λ-module Q and for any given projective resolution defining a

tth syzygy $\Omega^t_\Lambda(M)$. Since an infinitesimal deformation of a Γ-module is a lifting to Γ_2 (see [11]), weak liftability of a Γ-module corresponds to that the module has an infinitesimal deformation. Hence, if M in mod Γ has an infinitesimal deformation and gl.dim $\Lambda < \infty$, then $\mathrm{pd}_\Gamma M < \infty$. We also show if M in mod Γ has an infinitesimal deformation, then the completion of M with respect to the topology induced by the $\underline{m}$-adic topology also has an infinitesimal deformation as a module over the completion of Γ. It is not obvious that the notions of weakly liftable and liftable are different, but we give some examples showing that they are indeed different.

Section 4 is devoted to studying liftings and weak liftings in the situation $R \to R/(x) = \bar{R}$, where R is a commutative local Gorenstein ring and x an R-regular element in the maximal ideal of R. Let CM(R) denote the category of all finitely generated maximal Cohen–Macaulay modules over R. Here, we only consider (weak) liftability of maximal Cohen–Macaulay modules over $\bar{R}$. One of the main aims in this section is to characterize liftable and weakly liftable modules in CM$(\bar{R})$. We show that every lifting of a module C in $CM(\bar{R})$ is a submodule of the minimal Cohen–Macaulay approximation X_C of C over R and use this to characterize the liftable modules. Further, we prove that a module C in CM$(\bar{R})$ is weakly liftable to R if and only if $X_c/xX_C \simeq C \oplus \Omega_{\bar{R}}^{-1}(C)$. We prove that the category of liftable and also the category of weakly liftable modules to R in CM$(\bar{R})$ are closed under syzygies, cosyzygies, and taking duals. Finally, we show that the category of weakly liftable modules to R in CM$(\bar{R})$ is functorially finite in mod $\bar{R}$ (see [4] or Section 4 for definition of a functorially finite subcategory).

Also in the last section we study weak liftings over a commutative local Gorenstein ring R, but now we consider all the finitely generated modules over $\bar{R} = R/(x)$ for an R-regular element x in the maximal ideal of R. We use the definition of a weak lifting of a module given in Section 3 and this section is mainly devoted to studying the properties of weakly liftable $\bar{R}$-modules in terms of their minimal Cohen–Macaulay approximation over R.

1. Lifting

Throughout this section let Λ be a noetherian R-algebra over a commutative complete local ring R and let $\underline{m}$ denote the maximal ideal in R. Let $\{x_1, x_2, \ldots, x_t\}$ be a Λ-regular sequence in $\underline{m}$ and $\Gamma = \Lambda/(x_1, x_2, \ldots, x_t)\,\Lambda$. This section is devoted to developing a obstruction theory for lifting modules over Γ to Λ.

We start by recalling the definition of a lifting of a module over arbitary rings.

DEFINITION. Let $\Delta \to \Sigma$ be a homomorphism of rings and let M be in mod Σ. Then L in mod Δ is called a *lifting* of M to Δ if the following two conditions are satisfied

(a) $M \simeq \Sigma \otimes_{\Delta} L$;

(b) $\operatorname{Tor}_i^{\Delta}(\Sigma, L) = (0)$ for all $i > 0$.

The Σ-module M is said to be *liftable* to Δ, when such a Δ-module L exists.

Remark. Let Λ be an R-order, where R is a discrete valuation ring and $0 \neq x \in \underline{m}$. If $\Gamma = \Lambda/x\Lambda$, then it is easy to see that $\operatorname{Tor}_i^{\Lambda}(\Gamma, L) = (0)$ for $i > 0$ for a Λ-module L if and only if x is L-regular. Since x is L-regular if and only if L is a free R-module (L is a Λ-lattice), the generalized notion of lifting specializes to the usual notion of lifting.

From now on let Λ be a noetherian R-algebra, where R is a commutative complete local ring and x a Λ-regular element, although some of the results hold true in a more general setting. Since Λ is a finitely generated R-module, Λ is complete with respect to the $\underline{m}$-adic topology and therefore also complete with respect to the topology induced by any proper ideal of R. Let $\Lambda_i = \Lambda/x^i\Lambda$ for all $i \geqslant 1$. We will now restrict ourselves to studying the lifting problem in the following cases, $\Lambda \to \Lambda_1$ and $\Lambda_i \to \Lambda_1$.

Our first aim is to show that if $\operatorname{Ext}^2_{\Lambda_1}(M, M) = (0)$ for a Λ_1-module M, then M is liftable to Λ. The proof consists of several steps. First we characterize liftable Λ_1-modules in terms of a lifting sequence and then give obstructions for lifting in terms of elements in $\operatorname{Ext}^2_{\Lambda_1}(M, M)$.

In order to prove the characterization of liftable Λ_1-modules in terms of a lifting sequence we need the following lemma.

LEMMA 1.1. (a) *For M in* mod Λ *we have*

$$\operatorname{Tor}_k^{\Lambda}(\Lambda_1, M) = \begin{cases} \{m \in M \mid xm = 0\}, & k = 1 \\ 0, & k > 1 \end{cases}.$$

(b) *For M in* mod Λ_i *we have*

$$\operatorname{Tor}_k^{\Lambda_i}(\Lambda_1, M) = \begin{cases} \{m \in M \mid xm = 0\}/x^{i-1}M, & k \text{ odd} \\ \{m \in M \mid x^{i-1}m = 0\}/xM, & k \text{ even} \end{cases}.$$

Proof. Using that $0 \to \Lambda \xrightarrow{x} \Lambda \to \Lambda_1 \to 0$ is free resolution of Λ_1 over Λ the proof of (a) is easy. Similarly, observing that

$$\cdots \Lambda_i \xrightarrow{x^{i-1}} \Lambda_i \xrightarrow{x} \Lambda_i \xrightarrow{x^{i-1}} \Lambda_i \xrightarrow{x} \Lambda_i \longrightarrow \Lambda_1 \longrightarrow 0$$

is a free resolution of Λ_1 over Λ_i the statement in (b) follows easily.

Let M in mod Λ_1 be liftable to Λ with lifting L and let $L_n = L/x^nL$. It is easy to see that L_n in mod Λ_n is a lifting of L_{n-1} to Λ_n for all $n \geqslant 2$. In fact, such a sequence $L_1 = M$, L_2, L_3, ... characterizes liftable Λ-modules.

THEOREM 1.2. *The following are equivalent for M in* mod Λ_1.

(a) *M is liftable to Λ.*

(b) *There exists a sequence $L_1 \simeq M$, L_2, L_3, ... of Λ-modules such that L_{n+1} in* mod Λ_{n+1} *is a lifting of L_n in* mod Λ_n *to Λ_{n+1} for all $n \geqslant 1$.*

Proof. The proof of the fact that (a) implies (b) is already given and this implication is true even in a more general setting.

Assume that there is a sequence $L_1 \simeq M$, L_2, L_3, ... of Λ-modules such that L_{n+1} in mod Λ_{n+1} is a lifting of L_n in mod Λ_n to Λ_{n+1} for all $n \geqslant 1$. Since L_{n+1} is a lifting of L_n to Λ_{n+1}, it is not hard to see that L_{n+1} is a lifting of M to Λ_{n+1}. Then it follows from Lemma 1.1(b) that we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{n+1}/x^nL_{n+1} & \xrightarrow{\bar{x}} & L_{n+1} & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{p}_n^{n+1}} & & \downarrow{\scriptstyle p_n^{n+1}} & & \| & & , \\ 0 & \longrightarrow & L_n/x^{n-1}L_n & \xrightarrow{\bar{x}} & L_n & \longrightarrow & M & \longrightarrow & 0(*) \end{array}$$

where the maps p_n^{n+1} and $\bar{p}_n^{n+1}$ are the epimorphisms induced by the epimorphism $L_{n+1} \to \Lambda_n \otimes_{\Lambda_{n+1}} L_{n+1} \simeq L_{n+1}/x^nL_{n+1} \simeq L_n$. Then the modules L_n and the maps $p_n^{n+1}: L_{n+1} \to L_n$ define an inverse system. Similarly we have that $L_n/x^{n-1}L_n$ and $\bar{p}_n^{n+1}: L_{n+1}/x^nL_{n+1} \to L_n/x^{n-1}L_n$ define an inverse system and it is not hard to see that they are isomorphic. Since $\{L_n, p_n^m\}$ is a surjective inverse system, the sequence $(*)$ remains exact when passing to the inverse limit by [5, Prop. 10.2, p. 104]. Let L denote $\varprojlim_n L_n$. We obtain the exact sequence of Λ-modules

$$0 \longrightarrow L \xrightarrow{x} L \longrightarrow M \longrightarrow 0,$$

since evidently multiplication by x is the unique Λ-homomorphism commuting with the maps $L_{n-1} \xrightarrow{x} L_n$. According to the definition of a lifting, L is a lifting of M to Λ if L is a finitely generated Λ-module, which we will prove next.

We could use the same method of proof as above replacing M with L_n and x with x^n and obtain the following exact sequence $0 \to L \xrightarrow{x^n} L \to L_n \to 0$. Hence, we have $L/x^nL \simeq L_n$. Since M is a finitely generated Λ_1-module, there exists an epimorphism $\pi: \Lambda^t \to M$ for a finite t. We then have the following commutative diagram:

$$\begin{tikzcd}
 & \Lambda' \arrow[dl, "\varphi"'] \arrow[d, "\pi"] & \\
L \arrow[r] \arrow[d, "\alpha_n"] & M \arrow[r] \arrow[d, equal] & 0. \\
L_n \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Using that L_n is a finitely generated R-module and the Nakayama Lemma, it follows that $\varphi_n = \alpha_n \circ \varphi$ is an epimorphism for every n. Therefore the induced map $\bar{\varphi}_n \colon \Lambda'/x^n\Lambda' \to L_n$ is also an epimorphism for every n. These maps induce the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & x^{n-1}\Lambda'/x^n\Lambda' \arrow[r, "s_n"] \arrow[d] & x^{n-1}L_n \arrow[d] & \\
0 \arrow[r] & K_n \arrow[r] \arrow[d, "r^n_{n-1}"] & \Lambda'/x^n\Lambda' \arrow[r, "\bar{\varphi}_n"] \arrow[d, "t^n_{n-1}"] & L_n \arrow[r] \arrow[d, "p^n_{n-1}"] & 0. \\
0 \arrow[r] & K_{n-1} \arrow[r] & \Lambda'/x^{n-1}\Lambda' \arrow[r, "\bar{\varphi}_{n-1}"] \arrow[d] & L_{n-1} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $\bar{\varphi}_n$ is an epimorphism, it follows easily that s_n is an epimorphism and therefore by the Snake Lemma r^n_{n-1} is also an epimorphism. The exact sequences above define an exact sequence of inverse systems $0 \to \{K_n, r^m_n\} \to \{\Lambda'/x^n\Lambda', t^m_n\} \to \{L_n, p^m_n\} \to 0$, where $\{K_n, r^m_n\}$ is a surjective inverse system. Again, by [5, Prop. 10.2, p. 104] the sequence remains exact passing to the inverse limit and we get an epimorphism $\varprojlim_n \Lambda'/x^n\Lambda' = \Lambda' \to L$, since Λ is complete with respect to the x-adic topology. Hence, L is a finitely generated Λ-module and therefore L is a lifting of M to Λ.

Theorem 1.2 gives us a possible way to construct liftings of M in mod Λ_1 by constructing a lifting sequence $L_1 \simeq M, L_2, L_3, \ldots$. When considering such a lifting sequence the following observation is useful. A module L in mod Λ_i is a lifting of M in mod Λ_1 to Λ_i by Lemma 1.1(b) if and only if (a) $M \simeq L/xL$, (b) $\{l \in L \mid xl = 0\} = x^{i-1}L$ and (c) $\{l \in L \mid x^{i-1}l = 0\} = xL$. It is easy to verify that the following lemma is true.

LEMMA 1.3. *For L in* mod Λ_i, *the set* $\{l \in L \mid x^{i-1}l = 0\}$ *equals* xL *if and only if* $\{l \in L \mid xl = 0\}$ *equals* $x^{i-1}L$.

Now we will describe obstructions for constructing such a lifting sequence mentioned above in terms of elements in $\mathrm{Ext}^2_{\Lambda_1}(M, M)$. Assume that L in mod Λ_i is a lifting of M in mod Λ_1 to Λ_i. The exact sequence $0 \to \Omega_\Lambda(L) \to P \to L \to 0$ induces the following exact sequence by tensoring it by $\Lambda_1 \otimes_\Lambda -$

$$0 \to \mathrm{Tor}_1^\Lambda(\Lambda_1, L) \to \Omega_\Lambda(L)/x\Omega_\Lambda(L) \to P/xP \to M \to 0.$$

Since L is a lifting of M to Λ_i, then $\{l \in L \mid x^{i-1}l = 0\} = xL$ and $\{l \in L \mid x^l = 0\} = x^{i-1}L$ by Lemma 1.1. Since $\mathrm{Tor}_1^\Lambda(\Lambda_1, L) = \{l \in L \mid xl = 0\}$ by Lemma 1.1 and L is a Λ_i-lifting of M, it follows easily that $\mathrm{Tor}_1^\Lambda(\Lambda_1, L) = x^{i-1}L \simeq M$. Hence, we have the element in $\mathrm{Ext}^2_{\Lambda_1}(M, M)$,

$$0 \to M \to \Omega_\Lambda(L)/\Omega_\Lambda(L) \to P/xP \to M \to 0,$$

which we will denote by θ_L. Since $\mathrm{Ext}^2_{\Lambda_1}(M, M) \simeq \mathrm{Ext}^1_{\Lambda_1}(\Omega_{\Lambda_1}(M), M)$ and $P/xP \to M$ is a projective cover, the element θ_L corresponds to the element, θ'_L, $0 \to M \to \Omega_\Lambda(L)/x\Omega_\Lambda(L) \to \Omega_{\Lambda_1}(M) \to 0$ in $\mathrm{Ext}^1_{\Lambda_1}(\Omega_{\Lambda_1}(M), M)$. Whether θ_L (or θ'_L) is zero or not will give an obstruction for lifting L to Λ_{i+1}. But to prove this we need the following lemma.

LEMMA 1.4. *Let L' in* mod Λ_i *be a lifting of M to Λ_i. A module L in* mod Λ_{i+1} *is lifting of L' to Λ_{i+1} if and only if there is an exact sequence*

$$0 \to M \to L \to L' \to 0,$$

where $L/xL \simeq M$.

Proof. Assume that L in mod Λ_{i+1} is a lifting of L' in mod Λ_i to Λ_{i+1}. Since $\mathrm{Tor}_{2n+1}^{\Lambda_{i+1}}(\Lambda_i, L) = \{l \in L \mid x^i l = 0\}/xL$ and $\mathrm{Tor}_{2n}^{\Lambda_{i+1}}(\Lambda_i, L) = \{l \in L \mid xl = 0\}/x^iL$ and L is a Λ_{i+1}-lifting of L', we have the following exact sequences $0 \to xL \to L \xrightarrow{x^i} x^iL \to 0$ and $0 \to x^iL \to L \xrightarrow{x} xL \to 0$. Hence, $xL \simeq L/x^iL \simeq L'$ and $x^iL \simeq L/xL \simeq L'/xL' \simeq M$, so we have an exact sequence

$$0 \to M \to L \to L' \to 0,$$

where $L/xL \simeq M$.

Assume that there is an exact sequence $0 \to M \xrightarrow{g} L \xrightarrow{f} L' \to 0$ where $L/xL \simeq M$ and L' in mod Λ_i is a lifting of M to Λ_i. Then we have the following commutative diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \ker f|_{xL} \arrow[r] \arrow[d] & M \arrow[d, "g"] & & \\
0 \arrow[r] & xL \arrow[r] \arrow[d, "f"] & L \arrow[r] \arrow[d, "f"] & L/xL \arrow[r] \arrow[d, "\bar{f}"] & 0. \\
0 \arrow[r] & xL' \arrow[r] \arrow[d] & L' \arrow[r] \arrow[d] & L'/xL' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Since $\bar{f}: M \simeq L/xL \to L'/xL' \simeq M$ is a surjective endomorphism of a noetherian module, $\bar{f}$ is an isomorphism. We want to show that $x^iL = \{l \in L \mid xl = 0\}$. Assume that $f(xl) = xf(l) = 0$. Since L' is a Λ_i-lifting of M, $f(l) = x^{i-1}l_1$ for some $l_1$in L'. There exists l' in L such that $f(l') = l_1$. Then $f(l - x^{i-1}l') = 0$ and therefore $l - x^{i-1}l' = g(m)$ for some $m \in M$. By multiplying this equality with x, we have $xl = x^il'$ and therefore ker $f|_{xL} \subset x^iL$. Since $x^iL \subset \ker f|_{xL}$, we have $\ker f|_{xL} = x^iL = \operatorname{Im} g \simeq M$. Now, consider the following commutative diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \{l \in L \mid xl = 0\} \arrow[r] \arrow[d] & \{l \in L' \mid xl' = 0\} \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "g"] \arrow[d, "0"] & L \arrow[r, "f"] \arrow[d, "x"] & L' \arrow[r] \arrow[d, "x"] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & xL \arrow[r, "f"] \arrow[d] & xL' \arrow[r] \arrow[d] & 0 \\
 & M \arrow[r] \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}
.
$$

Since L' is a Λ_i-lifting of M, we have that $\{l \in L' \mid xl = 0\} = x^{i-1}L' \simeq M$. By the Snake Lemma the connecting homomorphism $\partial: M \simeq \{l \in L' \mid xl = 0\}$

$\to M$ is a surjective endomorphism of the noetherian module M, so ∂ is an isomorphism. Hence, $\{l \in L \mid xl = 0\} = \operatorname{Im} g = x^i L$ and by Lemma 1.3 $\operatorname{Tor}_k^{\Lambda_{i+1}}(\Lambda_i, L) = (0)$ for all $k > 0$. Since $\operatorname{Im} g = x^i L$, we have $L/x^i L \simeq L'$ and L is a lifting of L' to Λ_{i+1}.

Now we are ready to prove that whether θ_L is zero or not for a Λ_i-lifting L of M gives an obstruction for lifting L to Λ_{i+1}.

PROPOSITION 1.5. *Let M be in* mod Λ_1 *and assume that L in* mod Λ_i *is a lifting of M to Λ_i. Then the following are equivalent:*

(a) $\theta_L = 0$ *in* $\operatorname{Ext}^2_{\Lambda_1}(M, M)$.

(b) $\Omega_\Lambda(L)/x\Omega_\Lambda(L) \simeq M \oplus \Omega_{\Lambda_1}(M)$.

(c) *L is liftable to Λ_{i+1}.*

Proof. We have already remarked that the element θ_L in $\operatorname{Ext}^2_{\Lambda_1}(M, M)$ corresponds to the element $\theta'_L: 0 \to M \to \Omega_\Lambda(L)/x\Omega_\Lambda(L) \to \Omega_{\Lambda_1}(M) \to 0$ in $\operatorname{Ext}^1_{\Lambda_1}(\Omega_{\Lambda_1}(M), M)$. The element $\theta_L = 0$ if and only if $\theta'_L = 0$, that is, if and only if $0 \to M \xrightarrow{\alpha} \Omega_\Lambda(L)/x\Omega_\Lambda(L) \to \Omega_{\Lambda_1}(M) \to 0$ is a split exact sequence. Since α is a split monomorphism if and only if $\Omega_\Lambda(L)/x\Omega_\Lambda(L) \simeq M \oplus \Omega_{\Lambda_1}(M)$ by [12], we have shown that (a) and (b) are equivalent.

(b)$\Rightarrow$(c). Assume that the non-zero map $\alpha: M \to \Omega_\Lambda(L)/x\Omega_\Lambda(L)$ in θ'_L is a split monomorphism with splitting $\beta: \Omega_\Lambda(L)/x\Omega_\Lambda(L) \to M$. Let γ be the composition $\Omega_\Lambda(L) \to \Omega_\Lambda(L)/x\Omega_\Lambda(L) \xrightarrow{\beta} M$. Then we have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & \Omega_\Lambda(L) \arrow[r] \arrow[d, "\gamma"] & P \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\
\eta: & 0 \arrow[r] & M \arrow[r] & E \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

and applying the functor $\Lambda_1 \otimes_\Lambda -$ the following diagram is induced:

$$\begin{tikzcd}
\theta_L: 0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "s"] & \Omega_\Lambda(L)/x\Omega_\Lambda(L) \arrow[r] \arrow[d, "\beta"] & P/xP \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
 & M \arrow[r, "t"] & M \arrow[r] & E/xE \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Since $\beta \circ \alpha = \mathrm{id}_M = t \circ s$, the map t must be an epimorphism and $E/xE \simeq M$. Since the exact sequence η satisfies all the requirements in Lemma 1.4, we have that E in mod Λ_{i+1} is a lifting of L to Λ_{i+1}.

(c)$\Rightarrow$(*a*). Assume that E in mod Λ_{i+1} is a lifting of L to Λ_{i+1}. By Lemma 1.4 we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_\Lambda(L) & \longrightarrow & P & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{f} & L & \longrightarrow & 0
\end{array},$$

where $E/xE \simeq M$. Applying the functor $\Lambda_1 \otimes_\Lambda -$ to the diagram above induces the following commutative diagram:

$$\begin{array}{ccccccccccc}
\theta_L\colon 0 & \longrightarrow & M & \xrightarrow{\alpha} & \Omega_\Lambda(L)/x\Omega_\Lambda(L) & \longrightarrow & P/xP & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow & & \| & & \\
 & & M & \xrightarrow{\delta} & M & \xrightarrow{0} & E/xE & \xrightarrow{\bar{f}} & M & \longrightarrow & 0
\end{array}.$$

The map $\bar{f}\colon E/xE \to M$ induced by f is a surjective endomorphism of a noetherian module. Therefore $\bar{f}$ is an isomorphism and for the same reason the map δ is an isomorphism. Since $\delta = \beta \circ \alpha$, the map α is a split monomorphism and therefore $\theta_L = 0$ in $\mathrm{Ext}^2_{\Lambda_1}(M, M)$.

Combining Theorem 1.2 and Proposition 1.5 we reach our first aim, namely, the following sufficient condition for liftability (see also [11]).

PROPOSITION 1.6. *Let M be in* mod Λ_1. *If* $\mathrm{Ext}^2_{\Lambda_1}(M, M) = (0)$, *then M is liftable to Λ.*

One immediate consequence of this result is that every Λ_1-module of projective dimension less or equal to 1 is liftable to Λ. Another observation is that the converse of this result is *not* true, due to the following counterexample.

Let G be a cyclic group of prime order p and R a discrete valuation ring with maximal ideal generated by p. If $\Lambda = RG$ and $x = p$, then x is a Λ-regular element, the ring Λ is isomorphic to $R[t]/(t^p - 1)$ and $\Lambda_1 = \Lambda/x\Lambda = R/pR[t]/(t-1)^p$. Let $S_i = R/pR[t]/(t-1)^i$ for $i = 1, 2, \ldots, p$, which are all the indecomposable Λ_1-modules. It is shown in [15; see Section 2] that S_1, S_{p-1} and S_p are liftable to Λ. Since $\Omega_{\Lambda_1}(S_1) = S_{p-1}$ and $0 \to S_1 \to \Lambda_1 \to \Omega_{\Lambda_1}(S_1) \to 0$ is a non-split exact sequence, $\mathrm{Ext}^2_{\Lambda_1}(S_1, S_1) \neq (0)$ but S_1 is liftable to Λ.

Our next aim is to generalize Proposition 1.6 to the situation $\Lambda \to \Lambda/(x_1, x_2, \ldots, x_t)\Lambda = \Gamma$, where $\{x_1, x_2, \ldots, x_t\}$ is a Λ-regular sequence.

PROPOSITION 1.7. *Let M be in* mod Γ. *If* $\mathrm{Ext}^2_\Gamma(M, M) = (0)$, *then M is liftable to Λ.*

Proof. Let $\Gamma_i = \Lambda/(x_1, x_2, \ldots, x_i)\Lambda$ for $i = 1, \ldots, t$ and let $\Gamma_0 = \Lambda$. Since $\mathrm{Ext}^2_{\Gamma_t}(M, M) = (0)$ and x_t is regular on Γ_{t-1}, there exists a lifting L_{t-1} of

M to Γ_{t-1} by Proposition 1.6. Applying $\mathrm{Hom}_{\Gamma_{t-1}}(L_{t-1}, \ )$ to the sequence $0 \to L_{t-1} \xrightarrow{x_t} L_{t-1} \to M \to 0$, it induces the following long exact sequence:

$$\cdots \longrightarrow \mathrm{Ext}^2_{\Gamma_{t-1}}(L_{t-1}, L_{t-1}) \xrightarrow{x_t} \mathrm{Ext}^2_{\Gamma_{t-1}}(L_{t-1}, L_{t-1}) \longrightarrow \mathrm{Ext}^2_{\Gamma_{t-1}}(L, M) \to \cdots .$$

Since L_{t-1} is a lifting of M, we have that $\mathrm{Ext}^i_{\Gamma_t}(M, M) \simeq \mathrm{Ext}^i_{\Gamma_{t-1}}(L_{t-1}, M)$, so $\mathrm{Ext}^2_{\Gamma_{t-1}}(L_{t-1}, M) = (0)$ and $x_t \cdot \mathrm{Ext}^2_{\Gamma_{t-1}}(L_{t-1}, L_{t-1}) = \mathrm{Ext}^2_{\Gamma_{t-1}}(L_{t-1}, L_{t-1})$. Since $\mathrm{Ext}^2_{\Gamma_{t-1}}(L_{t-1}, L_{t-1})$ is a finitely generated R-module and $x_t \in \underline{m}$, we have $\mathrm{Ext}^2_{\Gamma_{t-1}}(L_{t-1}, L_{t-1}) = (0)$ by the Nakayama Lemma. Hence, the module L_{t-1} can be lifted to Γ_{t-2}. Continuing this process, we construct a sequence of modules $L_t = M, L_{t-1}, L_{t-2}, ..., L_0$ where L_i is a lifting of L_{i+1} to Γ_i for $i = t-1, t-2, ..., 0$. It follows that $\Gamma_t \otimes_\Lambda L_0 \simeq M$, and L_0 is a lifting of M to Λ if $\mathrm{Tor}^\Lambda_i(\Gamma_t, L_0) = (0)$ for all $i \geq 1$. Assume that $\mathrm{Tor}^{\Gamma_j}_i(\Gamma_t, L_j) = (0)$ for all $i \geq 1$ and $j \geq r+1$. Let $\mathbf{P}^\cdot$ be a projective resolution of L_r over Γ_r. Since L_r is a lifting of L_{r+1} to Γ_r, we have that $\Gamma_{r+1} \otimes_{\Gamma_r} \mathbf{P}^\cdot$ is a projective resolution of L_{r+1} over Γ_{r+1}. Further, since $\Gamma_t \otimes_{\Gamma_r} \mathbf{P}^\cdot \simeq \Gamma_t \otimes_{\Gamma_{r+1}} (\Gamma_{r+1} \otimes_{\Gamma_r} \mathbf{P}^\cdot)$, it follows that $\mathrm{Tor}^{\Gamma_r}_i(\Gamma_t, L_r) \simeq \mathrm{Tor}^{\Gamma_{r+1}}_i(\Gamma_t, L_{r+1}) = (0)$ for $i \geq 1$. By induction on r we have $\mathrm{Tor}^\Lambda_i(\Gamma_t, L_0) = (0)$ for all $i \geq 1$, and L_0 is a lifting of M to Λ.

We end this section by proving some applications of Proposition 1.7 in module theory. Let Λ and Γ be as above, then the following remark follows immediately from the definition of a lifting of a module. If L in mod Λ is a lifting of M in mod Γ and $\mathbf{P}^\cdot$ is a Λ-projective resolution of L, then $\Gamma \otimes_\Lambda \mathbf{P}^\cdot$ is a Γ-projective resolution of M. Then, the following proposition is an immediate consequence of this remark and Proposition 1.7.

PROPOSITION 1.8. *Assume that* gl. dim $\Lambda < \infty$. *If* $\mathrm{Ext}^2_\Gamma(M, M) = (0)$ *for* M *in* mod Γ, *then* $pd_\Gamma M < \infty$.

The second and last application is connected to the Nakayama Conjecture: If Λ is a finite dimensional algebra over a field k and dom.dim $\Lambda = \infty$, that is, in a minimal injective resolution

$$0 \to \Lambda \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

for Λ, all the E_i are projective, then Λ is selfinjective. In [3], Auslander and Reiten made the following conjecture: If Λ is an artin algebra and M is a Λ-generator with $\mathrm{Ext}^i_\Lambda(M, M) = (0)$ for all $i \geq 1$, then M is projective. Then they showed that the Nakayama Conjecture holds if their conjecture holds for all artin algebras. The conjecture of Auslander and Reiten mentioned above makes sense for any ring. We consider this conjecture in our setting and it is not hard to see that it is equivalent to the following.

If Λ is a noetherian R-algebra over a complete commutative local ring R and $\operatorname{Ext}^i_\Lambda(M \oplus \Lambda, M \oplus \Lambda) = (0)$ for all $i \geq 1$, then M is projective. Then, the following proposition is an easy consequence of Proposition 1.8.

PROPOSITION 1.9. *Let Λ be a noetherian R-algebra of finite global dimension over a complete commutative local ring R and let $\Gamma = \Lambda/(x_1, x_2, ..., x_t)\Lambda$, where $\{x_1, x_2, ..., x_t\}$ is a Λ-sequence in the maximal ideal of R. If $\operatorname{Ext}^i_\Gamma(M \oplus \Gamma, M \oplus \Gamma) = (0)$ for M in $\operatorname{mod} \Gamma$ for all $i \geq 1$, then M is projective.*

Proof. Assume that $\operatorname{Ext}^i_\Gamma(M \oplus \Gamma, M \oplus \Gamma) = (0)$ for M in $\operatorname{mod} \Gamma$ for all $i \geq 1$. By Proposition 1.8 $\operatorname{pd}_\Gamma M < \infty$, since $\operatorname{gl.dim} \Lambda < \infty$ and $\operatorname{Ext}^2_\Gamma(M \oplus \Gamma, M \oplus \Gamma) = (0)$. Since $\operatorname{Ext}^i_\Gamma(M, \Gamma) = (0)$ for all $i \geq 1$, it follows that M is projective.

This raises the question whether all artin algebras Γ are a factor of a noetherian R-algebra Λ of finite global dimension over a complete commutative local ring R of the form $\Gamma = \Lambda/(x_1, x_2, ..., x_t)\Lambda$ where $\{x_1, x_2, ..., x_t\}$ is a Λ-regular sequence in the maximal ideal of R. All complete intersections and therefore every group ring kG where G is a finite abelian group, k a field and the characteristic of k divides the order of G are of this form. But the following example suggested by J. Alperin shows that not all group rings kG are such a factor. Let k be a field of characteristic two and let G be the semi-direct product of a (normal) elementary abelian 2-group of order eight extended by a group of order seven which acts faithfully on the abelian 2-group. Let $\Gamma = kG$ and k the trivial Γ-module. Then it is known that $\operatorname{Ext}^2_\Gamma(k, k) = (0)$ and that $\operatorname{pd}_\Gamma k = \infty$. By Proposition 1.8 it follows that Γ is not of the form $\Lambda/(x_1, x_2, ..., x_t)\Lambda$ for any noetherian R-algebra Λ of finite global dimension where R is a complete commutative local ring and $\{x_1, x_2, ..., x_t\}$ is a Λ-regular sequence in the maximal ideal of R.

2. REALIZATION AND UNIQUENESS OF LIFTINGS

Throughout this section Λ will be a noetherian R-algebra over a commutative complete local ring R with maximal ideal $\underline{m}$. Let x be a Λ-regular element in $\underline{m}$ and $\{x_1, x_2, ..., x_t\}$ a Λ-regular sequence in $\underline{m}$. We will also in this section restrict our attention to studying the lifting problem in the following cases, $\Lambda \to \Lambda/x\Lambda = \Lambda_1$, $\Lambda_i = \Lambda/x^i\Lambda \to \Lambda_1$, and $\Lambda \to \Gamma$, where $\Gamma = \Lambda/(x_1, x_2, ..., x_t)\Lambda$. Throughout this section a regular element or a regular sequence on a Λ-module is always assumed to be elements in $\underline{m}$.

This section is devoted to showing that every lifting of a liftable Λ_1-module M is a factor module of $\Omega_\Lambda(M)$ and to discussing which properties the category of Λ_1-modules liftable to Λ or Λ_i has. Here, $\Omega_\Lambda(M)$ denotes the first syzygy of M over Λ given by the projective cover of M over Λ. We also show that if M in mod Γ is liftable to Λ and $\operatorname{Ext}^1_\Gamma(M, M) = (0)$, then the lifting is unique up to isomorphism. This is done by first showing that if M in mod Λ_1 is liftable and $\operatorname{Ext}^1_{\Lambda_1}(M, M) = (0)$, then the lifting is unique up to isomorphism (see [11]). Finally, we discuss when every liftable Λ_1-module has a unique lifting up to isomorphism. In the following a module M is said to have a unique lifting or to be uniquely liftable if M has a unique lifting up to isomorphism.

The aim now is to show that every lifting of a liftable Λ_1-module M is a factor module of $\Omega_\Lambda(M)$. This will lead us to a characterization of a liftable Λ_1-module M in terms of existence of special submodules of $\Omega_\Lambda(M)$.

PROPOSITION 2.1. *If M in* mod Λ_1 *is liftable to Λ, then every lifting of M is a factor module of $\Omega_\Lambda(M)$.*

Proof. Assume that M in mod Λ_1 is liftable to Λ with lifting L. Then we have the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & L \arrow[d, "x"] & \\
0 \arrow[r] & \Omega_\Lambda(L) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_\Lambda(M) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & L \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}.$$

It follows immediately from the Snake Lemma that we have an exact sequence $0 \to \Omega_\Lambda(L) \to \Omega_\Lambda(M) \to L \to 0$, hence L is a factor module of $\Omega_\Lambda(M)$.

Before we characterize a liftable Λ_1-module M in terms of existence of special submodules of $\Omega_\Lambda(M)$, we need to recall some notions from

Section 1. The element $\theta_M: 0 \to M \to \Omega_\Lambda(M)/x\Omega_\Lambda(M) \to P/xP \to M \to 0$ in $\operatorname{Ext}^2_{\Lambda_1}(M, M)$ is induced by tensoring the exact sequence $0 \to \Omega_\Lambda(M) \to P \to M \to 0$ by $\Lambda_1 \otimes_\Lambda -$. The element $\theta'_M: 0 \to M \to \Omega_\Lambda(M)/x\Omega_\Lambda(M) \to \Omega_{\Lambda_1}(M) \to 0$ is the element corresponding to the element θ_M by the isomorphism $\operatorname{Ext}^2_{\Lambda_1}(M, M) \simeq \operatorname{Ext}^1_{\Lambda_1}(\Omega_{\Lambda_1}, (M), M)$.

PROPOSITION 2.2. *A module M in* mod Λ_1 *is liftable to Λ if and only if there exists $N \subset \Omega_\Lambda(M)$ such that x is a $\Omega_\Lambda(M)/N$-regular element and $\pi(N) \subsetneq \Omega_\Lambda(M)/x\Omega_\Lambda(M) \xrightarrow{\beta} \Omega_{\Lambda_1}(M)$ is an isomorphism, where $\pi: \Omega_\Lambda(M) \to \Omega_\Lambda(M)/x\Omega_\Lambda(M)$ is the natural epimorphism and β is the map $\Omega_\Lambda(M)/x\Omega_\Lambda(M) \to \Omega_{\Lambda_1}(M)$ in the exact sequence θ'_M.*

Proof. We will now only prove the "only if"-part, because the "if"-part will be stated as Lemma 2.3 for later use.

Assume that M is liftable to Λ with lifting L. Then we have seen that we have the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & L \arrow[d, "x"] & \\
0 \arrow[r] & \Omega_\Lambda(L) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_\Lambda(M) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & L \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Let $N = \Omega_\Lambda(L)$, then x is a $\Omega_\Lambda(M)/N$-regular element. Applying the functor $\Lambda_1 \otimes_\Lambda -$ to the above diagram the following commutative diagram is induced:

$$\begin{tikzcd}
 & 0 \arrow[r] & N/xN \arrow[r] \arrow[d, "t"] & P/xP \arrow[r] \arrow[d, equal] & L/xL \arrow[r] \arrow[d, equal] & 0 \\
\theta_M: 0 \arrow[r] & M \arrow[r] & \Omega_\Lambda(M)/x\Omega_\Lambda(M) \arrow[r] & P/xP \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Hence, $N/xN \simeq \Omega_{\Lambda_1}(M)$ and it is easy to see that $\beta \circ t$ is an isomorphism. From the commutative diagram

$$\begin{tikzcd} & N \arrow[r, hook] \arrow[d] & \Omega_\Lambda(M) \arrow[d, "\pi"] \\ 0 \arrow[r] & N/xN \arrow[r, "t"] & \Omega_\Lambda(M)/x\Omega_\Lambda(M) \end{tikzcd}$$

we see that $\pi(N) = t(N/xN)$ and therefore the composition

$$\pi(N) \subsetneq \Omega_\Lambda(M)/x\Omega_\Lambda(M) \xrightarrow{\beta} \Omega_{\Lambda_1}(M)$$

is an isomorphism.

The following lemma completes the proof of Proposition 2.2 and the lemma will be used in Section 4.

LEMMA 2.3. *Let E in* mod Λ *be such that x is an E-regular element and let $0 \to A \xrightarrow{\alpha} E/xE \xrightarrow{\beta} B \to 0$ be an exact sequence in* mod Λ_1. *Let $\pi: E \to E/xE$ be the natural epimorphism. If there exists $N \subset E$, such that x is an E/N-regular element and the composition $\pi(N) \subsetneq E/xE \xrightarrow{\beta} B$ is an isomorphism, then N is a lifting of B and E/N is a lifting of A.*

Proof. Applying the functor $\Lambda_1 \otimes_\Lambda -$ to the sequence $0 \to N \xrightarrow{i} E \to L \to 0$ induces the following commutative diagram where $L = E/N$:

$$\begin{tikzcd} & N \arrow[r, "i"] \arrow[d] & E \arrow[d, "\pi"] & & \\ 0 \arrow[r] & N/xN \arrow[r, "\bar{i}"] & E/xE \arrow[r] & L/xL \arrow[r] & 0 \end{tikzcd}.$$

Since $N/xN \simeq \operatorname{Im} \bar{i} = \operatorname{Im}(\pi \circ i) = \pi(N)$ and the composition $\pi(N) \subsetneq E/xE \xrightarrow{\beta} B$ is an isomorphism, $N/xN \simeq B$ and the map $\bar{i}$ is a split monomorphism. Hence, $E/xE \simeq N/xN \oplus L/xL \simeq B \oplus A$. Since Λ_1 is a Krull–Schmidt ring and $N/xN \simeq B$, it follows that $L/xL \simeq A$. This shows that N is a lifting of B and L is a lifting of A.

Our next aim is to study the category of Λ_1-modules liftable to Λ or Λ_i. First, we show that the category of Λ_1-modules liftable to Λ or Λ_i is closed under syzygies.

PROPOSITION 2.4. *If M in* mod Λ_1 *is liftable to $\Lambda(\Lambda_i)$, then $\Omega_{\Lambda_1}(M)$ is also liftable to Λ (Λ_i). Moreover, if L is a lifting of M to Λ (Λ_i), then $\Omega_\Lambda(L)$ $(\Omega_{\Lambda_i}(L))$ is a lifting of $\Omega_{\Lambda_1}(M)$ to Λ (Λ_i).*

Proof. We only prove the result for Λ, since the proof for Λ_i is similar. Assume that M is liftable to Λ with lifting L. Tensoring the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega_\Lambda(L) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega_\Lambda(M) \arrow[r] & P \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

with $\Lambda_1 \otimes_\Lambda -$ the following commutative diagram is induced:

$$\begin{tikzcd} 0 \arrow[r] & \Omega_\Lambda(L)/x\Omega_\Lambda(L) \arrow[r] \arrow[d] & P/xP \arrow[r] \arrow[d, equal] & L/xL \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Omega_{\Lambda_1}(M) \arrow[r] & P/xP \arrow[r] & M \arrow[r] & 0 \end{tikzcd}.$$

Hence $\Omega_\Lambda(L)/x\Omega_\Lambda(L) \simeq \Omega_{\Lambda_1}(M)$ and since $\Omega_\Lambda(L)$ is a submodule of P, the module $\Omega_\Lambda(L)$ is a lifting of $\Omega_{\Lambda_1}(M)$ to Λ.

As mentioned in Section 1 the lifting problem has been studied for group rings over a ring of p-adic integers in [15]. Because of Proposition 2.4 it may be tempting to think that the category of Λ_1-module liftable to Λ (or Λ_i) is closed under kernels of epimorphisms, cokernels of monomorphisms and summands, but using the results in [15] it is easy to see the following.

(1) The category of Λ_1-modules liftable to Λ (or Λ_i) is not closed under kernels of epimorphisms or cokernels of monomorphisms.

(2) The category of Λ_1-modules liftable to Λ (or Λ_i) is not closed under extensions.

(3) The category of Λ_1-modules liftable to Λ is not closed under summands.

In the rest of this section we will study uniqueness of liftings. Let $\Gamma = \Lambda/(x_1, x_2, ..., x_t)\Lambda$ where $\{x_1, x_2, ..., x_t\}$ is a Λ-regular sequence. The first aim is to show that if M in $\operatorname{mod}\Gamma$ is liftable to Λ and $\operatorname{Ext}^1_\Gamma(M, M) = (0)$, then the lifting of M is unique. In order to prove this sufficient conditions for unique liftability we first consider the problem in the situation $\Lambda \to \Lambda_1$.

PROPOSITION 2.5. *If M in* $\operatorname{mod}\Lambda_1$ *is liftable and* $\operatorname{Ext}^1_{\Lambda_1}(M, M) = (0)$, *then the lifting of M to Λ is unique.*

Proof. Let L and L' be two liftings of M to Λ. The exact sequence $0 \to L \xrightarrow{x} L \to M \to 0$ induces the following long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}_\Lambda(L', L) \xrightarrow{x} \operatorname{Hom}_\Lambda(L', L) \longrightarrow \operatorname{Hom}_\Lambda(L', M) \\ &\longrightarrow \operatorname{Ext}^1_\Lambda(L', L) \xrightarrow{x} \operatorname{Ext}^1_\Lambda(L', L) \longrightarrow \operatorname{Ext}^1_\Lambda(L', M) \end{aligned}$$

Since L' is a lifting of M to Λ, we have $\operatorname{Ext}^1_\Lambda(L', M) \simeq \operatorname{Ext}^1_{\Lambda_1}(M, M) = (0)$ and therefore $\operatorname{Ext}^1_\Lambda(M', L) = x \cdot \operatorname{Ext}^1_\Lambda(L', L)$. Since $\operatorname{Ext}^1_\Lambda(L', L)$ is a finitely generated R-module and $x \in \underline{m}$, the Nakayama Lemma implies that $\operatorname{Ext}^1_\Lambda(L', L) = (0)$. Hence, the following sequence is exact:

$$0 \longrightarrow \operatorname{Hom}_\Lambda(L', L) \xrightarrow{x} \operatorname{Hom}_\Lambda(L', L) \longrightarrow \operatorname{Hom}_\Lambda(L', M) \longrightarrow 0.$$

This implies that we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & L' \arrow[r, "x"] \arrow[d] & L' \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "x"] \arrow[d] & L \arrow[r] \arrow[d, "g"] & M \arrow[r] \arrow[d, equal] & 0, \\ 0 \arrow[r] & L' \arrow[r, "x"] & L' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

where $\Lambda_1 \otimes_\Lambda g \circ f \colon L'/xL' \to L'/xL'$ is an isomorphism. Let $h = g \circ f$. Then the composition $L' \xrightarrow{h} L' \xrightarrow{\pi} L'/xL'$ is an epimorphism and therefore $xL'/\operatorname{Im} h = L'/\operatorname{Im} h$. Hence, h is an epimorphism by the Nakayama Lemma and h is an isomorphism since it is a surjective endomorphism of L'. Similarly, we can show that there are maps $f' \colon L \to L'$ and $g' \colon L' \to L$ such that the composition $g' \circ f'$ is an isomorphism. Since Λ has the Krull–Schmidt property, we have shown that L and L' are isomorphic.

Remark. The converse of this statement is not true. Let $R = \mathbf{Z}_{(3)}$ be the ring of 3-adic integers, G a cyclic group of order 3, $x = 3$ and $\Lambda = RG$. Then $\Lambda \simeq R[t]/(t^3 - 1)$ and $\Lambda_1 = (R/3R[t])/(t-1)^3 = \mathbf{Z}_3[t]/(t-1)^3$. Let $M = \mathbf{Z}_3[t]/(t-1)$, which is liftable to Λ and a lifting is $L = R[t]/(t-1)$. By [15, Lemma 2.1] every lifting of M is isomorphic to $R[t]/(Q)$, where Q is a monic polynomial of degree 1 dividing $t^3 - 1$ over R. Since $R[t]$ is a UFD, $t^3 - 1 = (t-1) \cdot (t^2 + t + 1)$ and $t - 1$ and $t^2 + t + 1$ are irreducible elements in $R[t]$, every lifting of M to Λ is isomorphic to L. But even though the lifting of M to Λ is unique $\operatorname{Ext}^1_{\Lambda_1}(M, M) \neq (0)$, since $0 \to M \to \mathbf{Z}_3[t]/(t-1)^2 \to M \to 0$ is a non-split exact sequence.

Let Γ denote $\Lambda/(x_1, x_2, ..., x_t)\Lambda$, where $\{x_1, x_2, ..., x_t\}$ is a Λ-regular sequence.

PROPOSITION 2.6. *If M in* mod Γ *is liftable and* $\operatorname{Ext}^1_\Gamma(M, M) = (0)$, *then the lifting of M to Λ is unique.*

Proof. Let L and L' in mod Λ be two liftings of M in mod Γ. Denote by Γ_i the R-algebra $\Lambda/(x_1, x_2, ..., x_i)\Lambda$ for all $i = 1, 2, ..., t$, by Γ_0 the R-algebra Λ and by L_r the Γ_r-module $\Gamma_r \otimes_\Lambda L$. We want to show that it is enough to show that L_r is a lifting of L_{r+1} to Γ_r for all $r = 0, 1, ..., t-1$ and similarly for L'. Then, since L_{t-1} and L'_{t-1} both are liftings of M to Γ_{t-1}, we have that $L_{t-1} \simeq L'_{t-1}$ by Proposition 2.5. Using the same method of proof as in Proposition 1.7 we can show that $\operatorname{Ext}^1_{\Gamma_{t-1}}(L_{t-1}, L_{t-1}) = (0)$ and similarly for L'_{t-1}. Continuing this process it follows that $L \simeq L'$.

It is clear that $\Gamma_{r+1} \otimes_{\Gamma_r} L_r \simeq L_{r+1}$, so we need to prove that $\operatorname{Tor}_i^{\Gamma_r}(\Gamma_{r+1}, L_r) = (0)$ for all $i \geqslant 1$. Applying the functor $-\otimes_\Lambda L$ to the exact sequence $0 \to \Gamma_{t-1} \xrightarrow{x_t} \Gamma_{t-1} \to \Gamma_t = \Gamma \to 0$, we get the following long exact sequence:

$$\cdots \longrightarrow \operatorname{Tor}_i^\Lambda(\Gamma_{t-1}, L) \xrightarrow{x_t} \operatorname{Tor}_i^\Lambda(\Gamma_{t-1}, L) \longrightarrow \operatorname{Tor}_i^\Lambda(\Gamma_t, L) \longrightarrow \cdots.$$

Since L is a lifting of M to Λ, we have that $\operatorname{Tor}_i^\Lambda(\Gamma_t, L) = (0)$ for all $i \geqslant 1$. Hence, it follows that $\operatorname{Tor}_i^\Lambda(\Gamma_{t-1}, L) = x_t \cdot \operatorname{Tor}_i^\Lambda(\Gamma_{t-1}, L)$ and therefore $\operatorname{Tor}_i^\Lambda(\Gamma_{t-1}, L) = (0)$ for all $i \geqslant 1$ by the Nakayama Lemma since $x_t \in \underline{m}$. By induction we can show that $\operatorname{Tor}_i^\Lambda(\Gamma_r, L) = (0)$ for all $i \geqslant 1$ and for all $r = 1, 2, ..., t$. If $\mathbf{P}^\bullet$ is a projective resolution of L over Λ, then $\Gamma_r \otimes_\Lambda \mathbf{P}^\bullet$ is a projective resolution of L_r over Γ_r, since $\operatorname{Tor}_i^\Lambda(\Gamma_r, L) = (0)$ for all $i \geqslant 1$. Since $\Gamma_t \otimes_\Lambda \mathbf{P}^\bullet \simeq \Gamma_t \otimes_{\Gamma_r} (\Gamma_r \otimes_\Lambda \mathbf{P}^\bullet)$, we have that $\operatorname{Tor}_i^{\Gamma_r}(\Gamma_t, L_r) \simeq \operatorname{Tor}_i^\Lambda(\Gamma_t, L) = (0)$ for all $i \geqslant 1$. Similar as for L we show that $\operatorname{Tor}_i^{\Gamma_r}(\Gamma_{r+1}, L_r) = (0)$ for all $i \geqslant 1$ and for all $r = 0, 1, ..., t-1$. Hence, we have shown that L_r is a lifting of L_{r+1} to Γ_r.

If Λ_1 is a semisimple ring, then Proposition 2.5 immediately implies that every lifting to Λ is unique. We end this section by characterizing when every liftable Λ_1-module has an unique lifting. First, we give a necessary condition.

PROPOSITION 2.7. *If every liftable Λ_1-module has a unique lifting to Λ, then $x \cdot \operatorname{Ext}^1_\Lambda(A, \;) = (0)$ for every A in* mod Λ *for which x is an A-regular element.*

Proof. Assume that every liftable Λ_1-module has a unique lifting to Λ. Let L be in mod Λ where x is a L-regular element and let $M = L/xL$. Since M is liftable to Λ and in particular liftable to Λ_2, we have that $\Omega_\Lambda(M)/x\Omega_\Lambda(M) \simeq M \oplus \Omega_{\Lambda_1}(M)$ by Proposition 1.5 and by Proposition 2.4 we have that $L \oplus \Omega_\Lambda(L)$ is a lifting of $\Omega_\Lambda(M)/x\Omega_\Lambda(M)$ to Λ. Since $\Omega_\Lambda(M)$ is clearly also a lifting of $\Omega_\Lambda(M)/x\Omega_\Lambda(M)$ and every lifting is unique, $\Omega_\Lambda(M) \simeq L \oplus \Omega_\Lambda(L)$. The sequence $0 \to L \xrightarrow{x} L \xrightarrow{\pi} M \to 0$ induces the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Ext}^1_\Lambda(L,\ ) \arrow[r, "x"] & \operatorname{Ext}^1_\Lambda(L,\ ) \arrow[r, "\partial"] \arrow[d, "\gamma"] & \operatorname{Ext}^2_\Lambda(M,\ ) \arrow[d, "\varphi_M"] & \\
\theta: & 0 \arrow[r] & \operatorname{Ext}^1_\Lambda(L,\ ) \arrow[r] & \operatorname{Ext}^1_\Lambda(\Omega_\Lambda(M),\ ) &
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r, "\operatorname{Ext}^2_\Lambda(\pi,\ )"] & \operatorname{Ext}^2_\Lambda(L,\ ) \arrow[r] \arrow[d, "\varphi_L"] & \cdots \\
\arrow[r, "\operatorname{Ext}^1_\Lambda(\Omega_\Lambda(\pi),\ )"] & \operatorname{Ext}^1(\Omega_\Lambda(L),\ ) \arrow[r] & 0
\end{tikzcd},$$

where the natural isomorphisms φ_M and φ_L are induced from the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
\delta: & 0 \arrow[r] & \Omega_\Lambda(L) \arrow[r, "\Omega_\Lambda(\pi)"] \arrow[d, equal] & \Omega_\Lambda(M) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "x"] & 0 \\
n: & 0 \arrow[r] & \Omega_\Lambda(L) \arrow[r] & P \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "\pi"] & 0. \\
 & & & M \arrow[r, equal] \arrow[d] & M \arrow[d] & \\
 & & & 0 & 0 &
\end{tikzcd}$$

Since $\Omega_\Lambda(M) \simeq L \oplus \Omega_\Lambda(L)$, the sequence δ is split exact by [12] and therefore the sequence θ is also a split exact sequence. Since $\operatorname{Ext}^2_\Lambda(\pi,\ )$ is an epimorphism, the induced map $\gamma: \operatorname{Ext}^1_\Lambda(L,\ ) \to \operatorname{Ext}^1_\Lambda(L,\ )$ is an epimorphism by the Snake Lemma. Since γ is an surjective endomorphism of a noetherian R-module, γ is an isomorphism and therefore the connecting homomorphism $\partial: \operatorname{Ext}^1_\Lambda(L,\ ) \to \operatorname{Ext}^2_\Lambda(M,\ )$ is a monomorphism. Hence, it follows that $x \cdot \operatorname{Ext}^1_\Lambda(L,\ ) = (0)$.

Remarks. (1) The converse statement is not true, due to the following counterexample. Let $R = \mathbf{Z}_{(2)}$ be the ring of 2-adic integers, G a cyclic group of order 2, $x=2$ and $\Lambda = RG$. Then $\Lambda \simeq R[t]/(t^2-1)$, the factor ring $\Lambda_1 = R/2R[t]/(t-1)^2 = \mathbf{Z}_2[t]/(t-1)^2$ and $x \cdot \mathrm{Ext}^1_\Lambda(A, \ ) = (0)$ for every A in mod Λ where x is an Λ-regular element. Let $M = \mathbf{Z}_2[t]/(t-1)$, which is an indecomposable Λ_1-module. Since $E = R[t]/(t+1) = \Lambda(t-1)$ and $D = R[t]/(t-1) = \Lambda(t+1)$ are liftings of M, and E and D are not isomorphic, every lifting with respect to x is not unique.

(2) The condition $x \cdot \mathrm{Ext}^1_\Lambda(A, \ ) = (0)$ for A in mod Λ where x is an Λ-regular element, can be reformulated in the following way (see [10]). The condition that $x \cdot \mathrm{Ext}^1_\Lambda(A, \ ) = (0)$ for A in mod Λ is equivalent to the fact that the map given by multiplication by x, $\Lambda \xrightarrow{x} \Lambda$, factors through a projective Λ-module.

(3) If we consider the proof of Proposition 2.7 again, we see that if the lifting of a liftable Λ_1-module M and the lifting of $\Omega_\Lambda(M)/x\Omega_\Lambda(M) \simeq M \oplus \Omega_{\Lambda_1}(M)$ to Λ are unique, then $x \cdot \mathrm{Ext}^1_\Lambda(L, \ ) = (0)$ for the lifting L of M to Λ. But, if the lifting of M to Λ is unique it does not imply that $x \cdot \mathrm{Ext}^1_\Lambda(L, \ ) = (0)$ for every lifting L of M to Λ. The following example shows this. Let R be a discrete valuation ring with maximal ideal generated by an element x and let k denote the residue class field $R/(x)$. Let Λ be the ring $\left(\begin{smallmatrix} R & 0 \\ R & R \end{smallmatrix}\right)$, then the element x is a Λ-regular element and $\Lambda_1 = \left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right)$. Let M be the Λ_1-module $\left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right)/\left(\begin{smallmatrix} 0 & 0 \\ k & 0 \end{smallmatrix}\right)$ which is a simple injective module. Therefore $\mathrm{Ext}^i_{\Lambda_1}(M, M) = (0)$ for $i = 1, 2$ and M is uniquely liftable to Λ. The Λ-module L given by $\left(\begin{smallmatrix} R & 0 \\ R & 0 \end{smallmatrix}\right)/\left(\begin{smallmatrix} 0 & 0 \\ R & 0 \end{smallmatrix}\right)$ is the lifting of M. Since the map given by multiplication by x, $L \xrightarrow{x} L$, does not factor through a projective Λ-module, $x \cdot \mathrm{Ext}^1_\Lambda(L, \ ) \neq (0)$. So, even if M is uniquely liftable, $x \cdot \mathrm{Ext}^1_\Lambda(L, \ )$ is not equal to (0) for the lifting L of M to Λ. Therefore we can also conclude that even if M in mod Λ is uniquely liftable, then $\Omega_\Lambda(M)/x\Omega_\Lambda(M) \simeq M \oplus \Omega_{\Lambda_1}(M)$ is not necessarily uniquely liftable to Λ. The Λ_1-module $M' = \Omega_{\Lambda_1}(M)$ given by $\left(\begin{smallmatrix} 0 & 0 \\ 0 & k \end{smallmatrix}\right)$ is a simple projective module. Then, $\mathrm{Ext}^i_{\Lambda_1}(M', M') = (0)$ for $i = 1, 2$ and therefore M' is uniquely liftable to Λ. So, even if M and M' are uniquely liftable to Λ, the direct sum $M \oplus M'$ is not uniquely liftable to Λ.

Our characterization of when every liftable Λ_1-module has a unique lifting uses a generalization of the Maranda Theorem [6]. For the convenience of the reader we recall the Maranda Theorem from [6]. Let Σ be an S-order in a finite dimensional separable K-algebra A, where S is a discrete valuation ring with prime element π and quotient ring K. Then there exists a nonnegative integer i_0 such that $\pi^{i_0} \cdot \mathrm{Ext}^1_\Sigma(M, N) = (0)$ for all Σ-lattices M and N [6, Coro. 29.5]. The Maranda Theorem reads as follows.

THEOREM 2.8 [6, Thm. 30.14]. *Let M and N be Σ-lattices and let i be a nonnegative integer. Then the following are true*:

(a) *If* $M \simeq N$ *as* Σ*-modules, then* $M/\pi^i M \simeq N/\pi^i N$ *as* $\Sigma/\pi^i\Sigma$*-modules for each* $i \geqslant 0$.

(b) *If* $M/\pi^i M \simeq N/\pi^i N$ *for some* $i \geqslant i_0 + 1$, *then* $M \simeq N$.

The Maranda Theorem gives examples when every liftable Λ_1-module has a unique lifting to Λ (when Λ is an order over a discrete valuation ring). The following generalization of the Maranda Theorem plays an important role in our characterization of when every liftable Λ_1-module has a unique lifting to Λ.

THEOREM 2.9. *Assume that* $x \cdot \mathrm{Ext}^1_\Lambda(E, E') = (0)$ *for all* Λ*-modules* E *and* E' *where* x *is regular on* E *and* E'. *Let* L *and* L' *be in* mod Λ *for which* x *is regular. If* $L/x^2L \simeq L'/x^2L'$, *then* $L \simeq L'$.

Proof. The proof follows along the lines of the proof for orders in [14, Theorem 1.1]. Let L and L' be Λ-modules where x is regular on L and L', such that $L/x^2L \simeq L'/x^2L'$. Let $\varphi: L/x^2L \to L'/x^2L'$ be an isomorphism and $f: P \to L'/x^2L'$ a projective cover of L'/x^2L' over Λ. By the Nakayama Lemma and since P is projective, there exists an epimorphism $g: P \to L$ such that $\pi \circ g = \varphi^{-1} \circ f$, where $\pi: L \to L/x^2L$ is the natural epimorphism. By [10] $x \cdot \mathrm{Ext}^1_\Lambda(L, \ ) = (0)$ if and only if $L \xrightarrow{x} L$ factors through a projective Λ-module. Since $L \xrightarrow{x} L$ factors through a projective Λ-module if and only if it factors through a projective cover of L, we have the following commutative diagram:

$$\begin{tikzcd}
& P \arrow[r, equal] \arrow[d] & P \arrow[r, equal] \arrow[d] & P \arrow[d] \\
L \arrow[r, "x"] \arrow[ur, "s"] & L \arrow[r, "\pi"] & L/x^2L \arrow[r, "\varphi"] & L'/x^2L'
\end{tikzcd} .$$

It follows from this diagram that the map $\psi = \varphi \circ \pi \circ x: L \to L'/x^2L'$ factors through a projective, hence there exists a map $\sigma: L \to L'$ such that $\rho \circ \sigma = \psi$ where $\rho: L' \to L'/x^2L'$ is the natural epimorphism. Since $\mathrm{Im}\, \psi = xL'/x^2L'$ and $\rho^{-1}(xL'/x^2L') = xL'$, the image $\mathrm{Im}\, \sigma \subset xL'$. By the Nakayama Lemma, $\mathrm{Im}\, \sigma = xL'$, since for every $xl' \in xL'$ there exists an $l \in L$ such that $xl' = \sigma(l) - x^2 l''$ for some $l'' \in L'$. Hence, $\sigma: L \to xL' \simeq L'$ is an epimorphism, since x is regular on L'. Assume that $\sigma(l) = 0$ for some $l \in L$. Since $\psi(l) = \rho \circ \sigma(l) = \varphi(\pi(xl))$ and φ is an isomorphism, $xl = x^2 l_1$ for some $l_1 \in L$. Since x is regular on L, we have $l = xl_1$ and $\sigma(l_1) = 0$. So by continuing this process the element $l \in \bigcap_{i=1}^{\infty} x^i L = (0)$, hence σ is a monomorphism and $L \simeq L'$.

Now the characterization of when every liftable Λ_1-module has a unique lifting to Λ follows easily from Proposition 2.7, Theorem 2.9 and [8, Thm. 2.2].

PROPOSITION 2.10. *Every liftable Λ_1-module has a unique lifting to Λ if and only if the following two conditions are satisfied*:

(a) $x \cdot \mathrm{Ext}^1_\Lambda(L, L') = (0)$ *for all Λ-modules L and L', where x is regular on L and L'.*

(b) *Every weakly Λ-liftable Λ_2-lifting of a Λ_1-module liftable to Λ is unique up to isomorphism.*

Proof. Assume that every liftable Λ_1-module has a unique lifting to Λ. Then by Proposition 2.7 we have $x \cdot \mathrm{Ext}^1_\Lambda(L, \ ) = (0)$ for every Λ-module L where x is regular on L.

Let M be liftable to Λ and let L_1 and L_2 be two Λ_2-liftings of M which are weakly liftable to Λ. Since weak liftability and liftability are the same for Λ_2-modules by [8, Thm. 2.2], the modules L_1 and L_2 are liftable to Λ. Then their liftings to Λ are also liftings of M to Λ and therefore L_1 and L_2 are isomorphic.

Assume that the conditions (a) and (b) are true. The condition (b) implies that if L and L' are two liftings of a Λ_1-module M, then L/x^2L and L'/x^2L' are two Λ_2-liftings of M and therefore $L/x^2L \simeq L'/x^2L'$. By condition (a) and the Maranda Theorem we have that $L \simeq L'$, hence every Λ_1-module liftable to Λ has a unique lifting.

3. WEAK LIFTING

Throughout this section let Λ be a noetherian R-algebra over a commutative local ring R with maximal ideal $\underline{m}$. Let $\{x_1, x_2, ..., x_t\}$ be a Λ-regular sequence of central elements in Λ and denote by Γ the factor ring $\Lambda/(x_1, x_2, ..., x_t)\Lambda$ and I the ideal $(x_1, x_2, ..., x_t)\Lambda$.

This section is mainly devoted to introducing and characterizing the notion of a weak lifting of a Γ-module. We show that the following are equivalent for M in mod Γ, (a) the module M is weakly liftable to Λ, (b) the module M is liftable to $\Lambda/I^2\Lambda$ and (c) the module M is isomorphic to a direct summand of $\Omega^t_\Lambda(M)/I\Omega^t_\Lambda(M) \oplus Q/IQ$ for a projective Λ-module Q and for any given projective resolution of M defining a tth syzygy $\Omega^t_\Lambda(M)$. We prove this result by first considering the case $t = 1$, which follows quite easily from Proposition 1.5 observing that we never used that R was complete and that the Λ-regular element x was an element in the maximal ideal of R.

Let $\Gamma_i = \Lambda/I^i\Lambda$ for all $i \geqslant 1$. In deformation theory a module L in mod Γ_2 is called an infinitesimal deformation of M in mod Γ if L is a Γ_2-lifting of

M (see [11]). By the result mentioned above a Γ-module has an infinitesimal deformation if and only if M has a weak lifting to Λ. This connection with infinitesimal deformations is one of the reasons for introducing the notion of a weak lifting of a module, but it also seems to be the right class of modules to consider in view of some of the problems studied in [15]. We define the notion of a weak lifting of a module in the same general context as for a lifting of a module in the following way.

DEFINITION. Let $\Delta \to \Sigma$ be a homomorphism rings and let M be in $\operatorname{mod} \Sigma$. Then, L in $\operatorname{mod} \Delta$ is called a *weak lifting* of M to Δ if the following two conditions are satisfied

(a) M is a direct summand of $\Sigma \otimes_{\Delta} L$;

(b) $\operatorname{Tor}_i^{\Delta}(\Sigma, L) = (0)$ for all $i > 0$.

The Σ-module M is said to be *weakly liftable* to Δ, if it has a weak lifting.

Every Γ-module liftable to Λ is obviously also weakly liftable to Λ. It is not obvious that these two notions of liftability are different, but we see later they indeed are different. An immediate consequence of the definition of weak liftability and our main result mentioned above is the following result.

PROPOSITION 3.1. *Assume that* $\operatorname{gl.dim} \Lambda < \infty$. *If* M *in* $\operatorname{mod} \Gamma$ *has an infinitesimal deformation, then* $\operatorname{pd}_{\Gamma} M < \infty$.

Examples of algebras Γ where this result applies are complete intersections and therefore also every group ring kG where G is a finite abelian group, k a field and the characteristic of k divides the order of G.

Our first step towards characterizing the weakly liftable Γ-modules is to consider the case $t = 1$. Let x be a central Λ-regular element in Λ and let $\Lambda_i = \Lambda/x^i\Lambda$ for all $i \geqslant 1$. Since we in this section do not assume that R is complete, we do not necessarily have projective covers. For M in $\operatorname{mod} \Lambda_1$ let $P \to M$ be an epimorphism where P is a projective Λ-module and denote the kernel by $\Omega_{\Lambda}(M)$. Tensoring the exact sequence $0 \to \Omega_{\Lambda}(M) \to P \to M \to 0$ with $\Lambda_1 \otimes_{\Lambda} -$ the following exact sequence is induced:

$$0 \to M \to \Omega_{\Lambda}(M)/x_1\Omega_{\Lambda}(M) \to P/x_1 P \to M \to 0.$$

This sequence represents an element in $\operatorname{Ext}^2 \Lambda_1(M, M)$ and observing that this element is independent of the choice of an epimorphism $P \to M$, we denote this element by θ_M as in Section 1. Then we have the following characterization of a Λ_1-module M weakly liftable to Λ.

PROPOSITION 3.2. *For M in* mod Λ_1 *the following are equivalent*

(a) *M is weakly liftable to Λ.*

(b) *$\Omega_\Lambda(M)/x\Omega_\Lambda(M) \simeq M \oplus \Omega_{\Lambda_1}(M)$, where $\Omega_{\Lambda_1}(M)$ is induced from the projective resolution defining $\Omega_\Lambda(M)$.*

(c) *M is liftable to Λ_2.*

Proof. Observing that we never used that R was complete and that x was an element in $\underline{m}$ in Proposition 1.5, the statements in (b) and (c) are equivalent by Proposition 1.5. Since (b) obviously implies (a), we only need to prove that (a) implies (b) or (c).

Assume that E is a weak lifting of M to Λ. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_\Lambda(M) & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \end{array}.$$

Applying the functor $\Lambda_1 \otimes_\Lambda -$, we obtain the following commutative diagram:

$$\begin{array}{lccccccccccc} \theta_M\colon & 0 & \longrightarrow & M & \longrightarrow & \Omega_\Lambda(M)/x\Omega_\Lambda(M) & \longrightarrow & P/xP & \longrightarrow & M & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow & & \| & & \\ \eta\colon & 0 & \longrightarrow & M & \longrightarrow & K/xK & \longrightarrow & E/xE & \xrightarrow{s} & M & \longrightarrow & 0 \end{array}.$$

This shows that θ_M and η represent the same element in $\operatorname{Ext}^2_{\Lambda_1}(M, M)$ and since s is a split epimorphism, θ_M is zero in $\operatorname{Ext}^2_{\Lambda_1}(M, M)$. The element θ_M is zero if and only if (b) or (c) is true by Proposition 1.5, hence we have shown that (a) implies (b).

Using this characterization of weakly liftable Λ_1-modules it is not hard to find examples of Λ_1-modules which are weakly liftable but not liftable.

(1) Let $R = \mathbf{Z}_{(3)}$ be the ring or p-adic integers, G a cyclic group of order 27, $x = 3$ and $\Lambda = RG$. Then $\Lambda \simeq R[t]/(t^{27} - 1)$ and $\Lambda_1 = \Lambda/x\Lambda = \mathbf{Z}_3[t]/(t-1)^{27}$. Then it is shown in [15] that the Λ_1-modules $\mathbf{Z}_3[t]/(t-1)^i$ for $i = 4, 5, 22, 23$ are liftable to Λ_2 but not liftable to Λ.

(2) Let S be a commutative local domain of dimension 1 with maximal ideal $\underline{m}$ and not a discrete valuation ring. We want to prove that the residue field $k \simeq S/\underline{m}$ is weakly liftable but not liftable with respect to any element in $\underline{m}\backslash\underline{m}^2$. Let x be in $\underline{m}\backslash\underline{m}^2$ and $S_1 = S/(x)$. Then x is S-regular and we have the exact sequence $0 \to k \xrightarrow{f} \underline{m}/x\underline{m} \to \underline{m}/(x) \to 0$,

where $f(1)=x$, $\underline{m}=\Omega_S(S/\underline{m})$ and $\underline{m}/(x)=\Omega_{S_1}(k)$. Since the composition $k \xrightarrow{f} \underline{m}/x\underline{m} \xrightarrow{\pi} \underline{m}/\underline{m}^2$ is nonzero, there is a map $g: \underline{m}/\underline{m}^2 \to k$ such that $g \circ \pi \circ f = \mathrm{id}_k$. Therefore f is a split monomorphism, so that $\underline{m}/x\underline{m} \simeq k \oplus \underline{m}/(x)$. Hence, k as an S_1-module is weakly liftable to S. It is not hard to see that that k is liftable if and only if S is a discrete valuation ring.

The aim now is to generalize Proposition 3.2 to the situation $\Lambda \to \Lambda/(x_1, x_2, ..., x_t)\Lambda = \Gamma$, where $\{x_1, x_2, ..., x_t\}$ is a Λ-regular sequence of central elements in Λ. Denote by I the ideal $(x_1, x_2, ..., x_t)\Lambda$ in Λ and by Γ_i the factor ring $\Lambda/I^i\Lambda$ for all $i \geq 1$. We want to show that M in mod Γ is weakly liftable to Λ if and only if M is liftable to Γ_2. When considering liftings of M to Γ_2 we must compute $\mathrm{Tor}_i^{\Gamma_i}(\Gamma, E)$ for $i \geq 1$ and for Γ_2-modules E. These Tor-groups are found in the following lemma.

LEMMA 3.3. *Let E be in* mod Γ_2. *Then* $\mathrm{Tor}_1^{\Gamma_2}(\Gamma, E) = \{(e_i) \in E^t \mid \sum_{i=1}^t x_i e_i = 0\}/IE^t$ *and* $\mathrm{Tor}_i^{\Gamma_2}(\Gamma, E) = (0)$ *for all $i \geq 1$ if and only if* $\mathrm{Tor}_1^{\Gamma_2}(\Gamma, E) = (0)$.

Proof. Using the Koszul complex as a free resolution of Γ over Λ and that the maps in the complex are given by multiplication by elements in I, it follows that $\mathrm{Tor}_1^{\Lambda}(\Gamma, \Gamma) = \Gamma^t$. Applying the functor $\Gamma \otimes_\Lambda -$ to the exact sequence $0 \to I \to \Lambda \to \Gamma \to 0$ it follows easily that $I/I^2 \simeq \Gamma^t$. Hence, Γ has the following free resolution $\mathbf{P}^\cdot$ over Γ_2

$$\mathbf{P}^\cdot : \cdots \longrightarrow \Gamma_2^{t^2} \xrightarrow{\oplus_{i=1}^t d} \Gamma_2^t \xrightarrow{(x_1, x_2, ..., x_t) = d} \Gamma_2 \longrightarrow \Gamma \longrightarrow 0.$$

Using this free resolution of Γ over Γ_2 it follows that $\mathrm{Tor}_1^{\Gamma_2}(\Gamma, E) = \{(e_i) \in E^t \mid \sum_{i=1}^t x_i e_i = 0\}/IE^t$. Since $\Omega^i_{\Gamma_2}(\Gamma) \simeq {}^{t^i}$, we have that $\mathrm{Tor}_i^{\Gamma_i}(\Gamma, E) = (0)$ for all $i \geq 1$ if and only if $\mathrm{Tor}_1^{\Gamma_2}(\Gamma, E) = (0)$ and the desired results follow directly.

Similar as for the case $t = 1$, we can give the following characterization of a lifting of M in mod Γ to Γ_2.

LEMMA 3.4. *A module E in* mod Γ_2 *is a lifting of M in* mod Γ *to* Γ_2 *if and only if there exists an exact sequence*

$$0 \longrightarrow M^t \longrightarrow E \xrightarrow{f} M \longrightarrow 0,$$

where E is in mod Γ_2 *and the map f induces an isomorphism $E/IE \to M$.*

Proof. Assume that E in mod Γ_2 is a lifting of M to Γ_2. Then we have the following exact sequence $0 \to IE \to E \to M \to 0$ and $\mathrm{Tor}_i^{\Gamma_2}(\Gamma, E) = (0)$

for all $i \geqslant 1$. Applying the functor $\Gamma \otimes_{\Gamma_2} -$ to this sequence induces the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^{\Gamma_2}(\Gamma, M) \longrightarrow IE \xrightarrow{0} E/IE \longrightarrow M \longrightarrow 0,$$

since $I^2E = (0)$. By Lemma 3.3 we have that $\operatorname{Tor}_1^{\Gamma_2}(\Gamma, M) = M'$ and hence $IE \simeq M'$. Therefore there exists an exact sequence $0 \to M' \to E \xrightarrow{f} M \to 0$, where E is in $\operatorname{mod} \Gamma_2$ and the map f induces an isomorphism $E/IE \to M$.

Assume that there exists an exact sequence $0 \to M' \to E \xrightarrow{f} M \to 0$, where E is in $\operatorname{mod} \Gamma_2$ and the map f induces an isomorphism $E/IE \to M$. In order to show that E is a lifting of M to Γ_2, it is sufficient to show that $\operatorname{Tor}_1^{\Gamma_2}(\Gamma, E) = (0)$ by Lemma 3.3. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & IE' \arrow[r] \arrow[d, equal] & K \arrow[r, "0"] \arrow[d] & IE \arrow[d] & \\
0 \arrow[r] & IE' \arrow[r] \arrow[d, "0"] & E' \arrow[r] \arrow[d, "d"] & M' \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & IE \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
 & IE \arrow[r] \arrow[d] & M \arrow[r, equal] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd},$$

where $d = (x_1, x_2, ..., x_t)$. By the Snake Lemma we have that $\ker d = IE'$. Since Γ has the following free resolution $\mathbf{P}^{\bullet}$ over Γ_2,

$$\mathbf{P}^{\bullet} : \cdots \longrightarrow \Gamma_2^{t^2} \xrightarrow{\oplus_{i=1}^{t} d} \Gamma_2^{t} \xrightarrow{(x_1, x_2, ..., x_t) = d} \Gamma_2 \longrightarrow \Gamma \longrightarrow 0,$$

the middle vertical sequence in the diagram above is the start of the complex $\mathbf{P}^{\bullet} \otimes_{\Gamma_2} E$. Since $IE' = \operatorname{Im}((\oplus_{i=1}^{t} d) \otimes_{\Gamma_2} E) \subseteq \ker d$, we have that $\operatorname{Im}((\oplus_{i=1}^{t} d) \otimes_{\Gamma_2} E) = \ker d$ and therefore $\operatorname{Tor}_1^{\Gamma_2}(\Gamma, E) = (0)$ and E is a lifting of M to Γ_2.

Let M be in $\operatorname{mod} \Gamma$. Similar to the case $t = 1$, tensoring the exact sequence $0 \to \Omega_\Lambda(M) \to P \to M \to 0$ with $\Gamma \otimes_\Lambda -$, the following exact sequence is induced:

$$0 \to \operatorname{Tor}_1^\Lambda(\Gamma, M) \to \Omega_\Lambda(M)/I\Omega_\Lambda(M) \to P/IP \to M \to 0.$$

Using the Koszul complex as a free resolution of Γ over Λ and that the maps in the complex are given by multiplication by elements in I, it follows that $\operatorname{Tor}_i^\Lambda(\Gamma, M) = M^{\binom{t}{i}}$ for $i = 1, 2, ..., t$ and (0) for $i > t$. Hence, the exact sequence above is

$$0 \to M' \to \Omega_\Lambda(M)/I\Omega_\Lambda(M) \to P/IP \to M \to 0,$$

which represents an element in $\operatorname{Ext}_\Gamma^2(M, M')$. Observing that this element is independent of the choice of an epimorphism $P \to M$, we denote it by θ_M. Now, we show that M in mod Γ is liftable to Γ_2 if and only if $\theta_M = 0$ in $\operatorname{Ext}_\Gamma^2(M, M')$.

PROPOSITION 3.5. *The following are equivalent for M in* mod Γ.

(a) $\theta_M = 0$ *in* $\operatorname{Ext}_\Gamma^2(M, M')$.

(b) *M is liftable to Γ_2.*

Proof. (a)$\Rightarrow$(b). Assume that $\theta_M = 0$ in $\operatorname{Ext}_\Gamma^2(M, M')$. Let $0 \to \Omega_\Lambda(M) \to P \to M \to 0$ be an exact sequence with P a projective Λ-module and denote the kernel of the induced epimorphism $P/IP \to M$ by $\Omega_\Gamma(M)$. Since $\operatorname{Ext}_\Gamma^2(M, M') \simeq \operatorname{Ext}_\Gamma^1(\Omega_\Gamma(M), M')$, the element $\theta_M = 0$ if and only if $\theta'_M : 0 \to M' \xrightarrow{\alpha} \Omega_\Lambda(M)/I\Omega_\Lambda(M) \to \Omega_\Gamma(M) \to 0$ is a split exact sequence. Let β be a splitting of α and let γ be the composition $\Omega_\Lambda(M) \to \Omega_\Lambda(M)/I\Omega_\Lambda(M) \xrightarrow{\beta} M'$. Then we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_\Lambda(M) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

and applying the functor $\Gamma \otimes_\Lambda -$, the following commutative diagram is induced:

$$\begin{tikzcd}
\theta_M : 0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, equal] & \Omega_\Lambda(M)/I\Omega_\Lambda(M) \arrow[r] \arrow[d, "\beta"] & P/IP \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
& M' \arrow[r, "r"] & M' \arrow[r] & E/IE \arrow[r] & M \arrow[r] & 0
\end{tikzcd}.$$

Since $\beta \circ \alpha = \mathrm{id}_M = r$, we have that $E/IE \simeq M$. Since E is in mod Γ_2, the module E is a lifting of M to Γ_2 by Lemma 3.4.

(b)$\Rightarrow$(a). Assume that E is a lifting of M to Γ_2. Applying the functor $\Gamma \otimes_\Lambda -$ to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_\Lambda(M) & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M' & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \end{array},$$

the following commutative diagram is induced:

$$\begin{array}{ccccccccc} \theta_M\colon 0 & \longrightarrow & M' & \xrightarrow{\alpha} & \Omega_\Lambda(M)/I\Omega_\Lambda(M) & \longrightarrow & P/IP & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow & & \| & & \\ & & M' & \xrightarrow{\delta} & M' & \xrightarrow{0} & E/IE & \xrightarrow{\sim} & M & \longrightarrow & 0 \end{array}.$$

The map $\delta\colon M' \to M'$ is a surjective endomorphism of a noetherian module, so δ is an isomorphism. Since $\delta = \beta \circ \alpha$ is an isomorhism, α is a split monomorphism and therefore $\theta_M = 0$ in $\mathrm{Ext}^2_\Gamma(M, M')$.

For M in mod Γ the obstructions to lifting M to Γ_2 are t elements in $\mathrm{Ext}^2_\Gamma(M, M)$. Similar obstructions have been studied by Nastold in [13]. It follows from the above proof that if M is liftable to Γ_2, then M' and therefore M are direct summands of $\Omega_\Lambda(M)/I\Omega_\Lambda(M)$. But since $\mathrm{Tor}^\Lambda_i(\Gamma, \Omega_\Lambda(M))$ is not necessarily zero for all $i \geqslant 1$ when $t > 1$, we can not conclude from this that M is weakly liftable to Λ as in the case $t = 1$. The next aim is to show that liftability to Γ_2 is equivalent to weak liftability to Λ.

PROPOSITION 3.6. *The following are equivalent for M in* mod Γ.

(a) *M is weakly liftable to Λ.*

(b) *M is isomorphic to a direct summand of $\Omega'_\Lambda(M)/I\Omega'_\Lambda(M) \oplus Q/IQ$ for a projective Λ-module Q and for any given projective resolution defining a tth syzygy $\Omega'_\Lambda(M)$.*

(c) *M is liftable to Γ_2.*

Proof. (a)$\Rightarrow$(c). Let L in mod Λ be a weak lifting of M to Λ. Applying the functor $\Gamma \otimes_\Lambda -$ to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_\Lambda(M) & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \xrightarrow{f} & M & \longrightarrow & 0 \end{array}$$

induces the following commutative diagram:

$$\begin{array}{ccccccccc}
\theta_m\colon 0 & \longrightarrow & M' & \longrightarrow & \Omega_\Lambda(M)/I\Omega_\Lambda(M) & \longrightarrow & P/IP & \longrightarrow & M & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \| & & \\
\eta\colon 0 & \longrightarrow & M' & \longrightarrow & K/IK & \longrightarrow & L/IL & \xrightarrow{\bar f} & M & \longrightarrow & 0
\end{array}$$

This shows that θ_M and η represent the same element in $\operatorname{Ext}^2_\Gamma(M, M')$ and since $\bar f$ is a split epimorphism, $\theta_M = 0$ in $\operatorname{Ext}^2_\Gamma(M, M')$. By Proposition 3.5 we have that M is liftable to Γ_2.

(c)$\Rightarrow$(b). Let E in $\operatorname{mod}\Gamma_2$ be a lifting of M to Γ_2 and let $E' = E/(x_2, ..., x_t)\, E$. Then $\Lambda/x_1\Lambda \otimes_{\Lambda/x_1^2\Lambda} E' \simeq \Gamma \otimes_\Lambda E$, so E' is a lifting of M viewed as a $\Lambda/x_1\Lambda$-module to $\Lambda/x_1^2\Lambda$, if $\operatorname{Tor}_i^{\Lambda/x_1^2\Lambda}(\Lambda/x_1\Lambda, E') = (0)$ for all $i \geqslant 1$. Since $\cdots \to \Lambda/x_1^2\Lambda \xrightarrow{x_1} \Lambda/x_1^2 \xrightarrow{x_1} \Lambda/x_1^2\Lambda \to \Lambda/x_1\Lambda \to 0$ is a free resolution of $\Lambda/x_1\Lambda$ over $\Lambda/x_1^2\Lambda$, we have that $\operatorname{Tor}^{\Lambda/x_1^2\Lambda}(\Lambda/x_1\Lambda, E') = \{e \in E' \mid x_1 e = 0\}/x_1 E'$. Let $e \in E$ and assume that $x_1 e = \sum_{i=2}^t x_i e_i$ for some $e_i \in E$. Since $\operatorname{Tor}_i^{\Gamma_2}(\Gamma, E) = (0)$ for all $i \geqslant 1$, we have that $\{(e_i) \in E' \mid \sum_{i=1}^t x_i e_i = 0\} = IE'$ by Lemma 3.3. Since $x_1 e - \sum_{i=2}^t x_i e_i = 0$, it follows that $e \in IE$ and therefore if $x_1 e' = 0$ for some $e' \in E'$, then $e' = x_1 e''$ for some $e'' \in E'$. This shows that $\{e \in E' \mid x_1 e = 0\} \subseteq x_1 E'$ and since the opposite inclusion always holds, we have shown that $\operatorname{Tor}_i^{\Lambda/x_1^2\Lambda}(\Lambda/x_1\Lambda, E') = (0)$ for all $i \geqslant 1$ and E' is a lifting of M to $\Lambda/x_1^2\Lambda$. By Proposition 3.2 we have that $\Omega_\Lambda(M)/x_1\Omega_\Lambda(M) \simeq M \oplus \Omega_{\Lambda/x_1\Lambda}(M)$. Since (c) implies (b) for $t = 1$ by Proposition 3.2, we want to finish the proof by induction on t.

Let $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M over Λ, then $\cdots \to P_2 \to P_1 \to \Omega_\Lambda(M) \to 0$ is a projective resolution of $\Omega_\Lambda(M)$ over Λ. Since x_1 is regular on $\Omega_\Lambda(M)$, we have that $\cdots \to P_2/x_1P_2 \to P_1/x_1P_1 \to \Omega_\Lambda(M)/x_1\Omega_\Lambda(M) \to 0$ is a projective resolution of $\Omega_\Lambda(M)/x_1\Omega_\Lambda(M)$ over $\Lambda/x_1\Lambda$. Since $\Omega_\Lambda(M)/x_1\Omega(M) \simeq M \oplus \Omega_{\Lambda/x_1\Lambda}(M)$ by Proposition 3.2, we have that $\Omega'^{-1}_{\Lambda/x_1\Lambda}(M)$ is a summand of $\Omega'_\Lambda(M)/x_1\Omega'_\Lambda(M) \oplus U$, where U is a projective $\Lambda/x_1\Lambda$-module by Schanuel's Lemma. Let $\Lambda' = \Lambda/x_1\Lambda$, then $\{x_2, ..., x_t\}$ is a Λ'-regular sequence and denote by Γ'_2 the factor ring $\Gamma_2/x_1\Gamma_2 = \Lambda'/(x_2, ..., x_t)^2\,\Lambda'$. Using that

$$\cdots \longrightarrow \Gamma_2^{t^2} \xrightarrow{\oplus_{i=1}^t d} \Gamma_2^t \xrightarrow{(x_1, x_2, ..., x_t) = d} \Gamma_2\,\Gamma_1 \xrightarrow{x_1} \Gamma_2 \longrightarrow \Gamma'_2 \longrightarrow 0$$

is a free resolution of Γ'_2 over Γ_2 and that $\operatorname{Tor}_i^{\Gamma_2}(\Gamma, E) = (0)$ for all $i \geqslant 1$, it is not hard to see that $\operatorname{Tor}_i^{\Gamma_2}(\Gamma'_2, E) = (0)$ for all $i \geqslant 1$. Therefore, if $\mathbf{P}^\bullet$ is a projective resolution of E over Γ_2, then $\Gamma'_2 \otimes_{\Gamma_2} \mathbf{P}^\bullet$ is a projective resolution of $\Gamma'_2 \otimes_{\Gamma_2} E \simeq E/x_1E$ over Γ'_2. Since $\Gamma \otimes_{\Gamma_2} \mathbf{P}^\bullet \simeq \Gamma \otimes_{\Gamma'_2} (\Gamma'_2 \otimes_{\Gamma_2} \mathbf{P}^\bullet)$ and $\Gamma \otimes_{\Gamma'_2} E/x_1E \simeq E/IE$, we have that $\operatorname{Tor}_i^{\Gamma'_2}(\Gamma, E/x_1E) \simeq \operatorname{Tor}_i^{\Gamma_2}(\Gamma, E) = (0)$ for all $i \geqslant 1$. Hence, E/x_1E is a lifting of M to Γ'_2 and by induction on t we have that M is a summand of $\Omega'^{-1}_{\Lambda'}(M)/(x_2, ..., x_t)\,\Omega'^{-1}_{\Lambda'}(M) \oplus Q'$, where

Q' is a projective Γ-module. Since $\Omega^{t-1}_{\Lambda'}(M)$ is a summand of $\Omega^t_\Lambda(M)/x_1\Omega^t_\Lambda(M) \oplus U$, we have that $\Omega^{t-1}_{\Lambda'}(M)/(x_2, ..., x_t)\,\Omega^{t-1}_{\Lambda'}(M) \oplus Q'$ is a direct summand of

$$\Omega^t_\Lambda(M)/x_1\Omega^t_\Lambda(M)/((x_2, ..., x_t)\,\Omega^t_\Lambda(M)/x_1\Omega^t_\Lambda(M))$$
$$\oplus\, U/(x_2, ..., x_t)\,U \oplus Q',$$

which is isomorphic to $\Omega^t_\Lambda(M)/I\Omega^t_\Lambda(M) \oplus U/(x_2, ..., x_t)\,U \oplus Q'$. Since $U/(x_2, ..., x_t)\,U \oplus Q'$ is a direct summand of Q/IQ for some projective Λ-module Q, the module M is isomorphic to a direct summand of $\Omega_\Lambda(M)/I\Omega_\Lambda(M) \oplus Q/IQ$ for some projective Λ-module Q.

(b) $\Rightarrow$ (a). Since $\operatorname{Tor}_i^\Lambda(\Gamma, \Omega^t_\Lambda(M) \oplus Q) = \operatorname{Tor}_{i+t}^\Lambda(\Gamma, M) = (0)$ for $i \geqslant 1$ and M is a direct summand of $\Omega^t_\Lambda(M)/I\Omega^t_\Lambda(M) \oplus Q/IQ$, it follows directly that M is weakly liftable to Λ.

Remark. If we assume that R is complete, the condition (b) in the Proposition 3.6 can be replaced with that M is a direct summand of $\Omega^t_\Lambda(M)/I\Omega^t_\Lambda(M)$, where the syzygies $\Omega^i_\Lambda(M)$ are defined by the minimal projective resolution of M over Λ.

Let $\hat{\Lambda}$ denote the completion of Λ with respect to the $\underline{m}$-adic topology. Then $\{x_1, x_2, ..., x_t\}$ is a $\hat{\Lambda}$-regular sequence of central elements in $\hat{\Lambda}$. Since the functor given by the completion with respect to the $\underline{m}$-adic topology is a full and faithful exact functor on mod Γ, the following result follows immediately from Proposition 3.5.

PROPOSITION 3.7. *A module M in* mod Γ *has an infinitesimal deformation if and only if $\hat{M}$ in* mod $\hat{\Gamma}$ *has.*

We end this section by studying some properties of the category of Γ-modules weakly liftable to Λ. Again, let x be a regular on Λ in $\underline{m}$ and $\Lambda_i = \Lambda/x^i\Lambda$ for all $i \geqslant 1$. Similar to when we consider the category of Λ_1-modules liftable to Λ, we can use the results in [15; see Section 2] to show that the category of Λ_1-modules weakly liftable to Λ is not closed under (a) kernels of epimorphisms, (b) cokernels of monomorphisms, and (c) extensions. But, contrary to the category of Λ_1-modules liftable to Λ the category of Λ_1-modules or Γ-modules weakly liftable to Λ is closed under summands, which is obvious from the definition a weakly liftable module. The category of Λ_1-modules or Γ-modules weakly liftable to Λ is as the category of Λ_1-modules liftable to Λ closed under syzygies, which we want to prove next. For the complete case and for the category of Λ_1-modules weakly liftable to Λ, this follows in two different ways from what we already have done. If M in mod Λ_1 is weakly liftable, then $\Omega_\Lambda(M)/x\Omega_\Lambda(M) \simeq M \oplus \Omega_{\Lambda_1}(M)$ by Proposition 3.2, hence $\Omega_{\Lambda_1}(M)$ is weakly liftable. Since M in mod Λ_1 is weakly liftable if and only if M is liftable to Λ_2, then $\Omega_{\Lambda_1}(M)$ is liftable to Λ_2 by Proposition 2.4 and therefore $\Omega_{\Lambda_1}(M)$ is weakly liftable to Λ if M is.

PROPOSITION 3.8. *If M in* mod Γ *is weakly liftable to Λ, then $\Omega_\Gamma(M)$ is also weakly liftable to Λ.*

Proof. Let L in mod Λ be a weak lifting of M to Λ. Applying the functor $\Gamma\otimes_\Lambda -$ to the exact sequence $0\to\Omega_\Lambda(L)\to P\to L\to 0$ induces the exact sequence $0\to\Omega_\Lambda(L)/I\Omega_\Lambda(L)\to P/IP\to L/IL\to 0$. Since M is a direct summand of L/IL, we have that $\Omega_\Gamma(M)$ is a direct summand of $\Omega_\Lambda(L)/I\Omega_\Lambda(L)\oplus U$, where U is a projective Γ-module by Schanuel's Lemma. Since U is a direct summand of Q/IQ for some projective Λ-module Q, it follows that $\Omega_\Gamma(M)$ is weakly liftable to Λ.

4. LIFTING, WEAK LIFTING, AND COHEN–MACAULAY MODULES

Throughout the rest of the paper R is a commutative local Gorenstein ring of dimension d and x is an R-regular element in the maximal ideal $\underline{m}$ of R. We denote by $CM(R)$ the category of all finitely generated maximal Cohen–Macaulay modules over R. We want to study the lifting problems for the situation $R\to R/(x)=\bar{R}$ and the modules we consider over $\bar{R}$ will be maximal Cohen–Macaulay modules. But before we briefly describe the results in this section, we will recall for the convenience of the reader some of the notions involved in this section.

The notion of Cohen–Macaulay approximations was introduced by M. Auslander and R.-O. Buchweitz in [2]. Given an R-module C, an exact sequence $0\to Y_C\to X_C\xrightarrow{f} C\to 0$ is called a right *Cohen–Macaulay approximation* of C if X_C is in $\mathrm{CM}(R)$ and $\mathrm{pd}_R\, Y_C<\infty$. An important property of these approximations is that every morphism $X\to C$ with X in $\mathrm{CM}(R)$ factors through f. The Cohen–Macaulay approximation is called *minimal* if f is a right minimal morphism, that is, an endomorphism $g: X_C\to X_C$ is an automorphism whenever $f=f\circ g$. The existence of Cohen–Macaulay and minimal Cohen–Macaulay approximations was established in [1, 2]. It was also shown that the minimal Cohen–Macaulay approximations are unique up to isomorphism.

Now, we define a lifting and a weak lifting in the setting in this section. A module L in $\mathrm{CM}(R)$ is said to be a *lifting* of M in $\mathrm{CM}(\bar{R})$ to R if $L/xL\simeq M$. If M is only isomorphic to a direct summand of L/xL, then L is said to be a weak lifting of M to R. The module M is then said to be *liftable* or *weakly liftable* to Λ, respectively. Since every module L in $\mathrm{CM}(R)$ is a submodule of a free R-module, every R-regular element is also

L-regular. So, the definition of a lifting above agrees with the definition given in Section 1. The definition of a weak lifting above does not agree with the definition in Section 1, but it follows easily from Proposition 3.2 that these two definitions are equivalent.

One of the main aims in this section is to characterize liftable and weakly liftable modules C in $\mathrm{CM}(\bar{R})$ in terms of their minimal Cohen–Macaulay approximation X_C over R and show that every lifting of a liftable module C in $\mathrm{CM}(\bar{R})$ is a submodule of X_C. We also study the category of modules in $\mathrm{CM}(\bar{R})$ liftable to R and the category of modules in $\mathrm{CM}(\bar{R})$ weakly liftable to R and show that they are closed under syzygies, cosyzgies, and taking duals. In addition, we show that the category of modules in $\mathrm{CM}(\bar{R})$ weakly liftable to R is functorially finite in mod $\bar{R}$, a notion we recall later in this section from [4]. Since we only deal with the lifting problems in the setting $R \to \bar{R}$, we denote by $\bar{L}$ the $\bar{R}$-module L/xL for any R-module L.

Our first aim is to show that every lifting of a liftable Cohen–Macaulay $\bar{R}$-module C is a submodule of the minimal Cohen–Macaulay approximation X_C in $\mathrm{CM}(R)$. This leads us to a characterization of liftability of C in $\mathrm{CM}(\bar{R})$ in terms of existence of special submodules of X_C.

PROPOSITION 4.1. *If C in $\mathrm{CM}(\bar{R})$ is liftable, then every lifting of C is a submodule of the minimal Cohen–Macaulay approximation of C of R.*

Proof. Assume that L in $\mathrm{CM}(R)$ is q lifting of C in $\mathrm{CM}(\bar{R})$. Then we have the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & \ker t \arrow[r] \arrow[d] & L \arrow[d, "x"] & \\
 & & L \arrow[r, equal] \arrow[d, "t"] & L \arrow[d] & \\
0 \arrow[r] & Y_C \arrow[r] & X_C \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

We want to show that $\ker t = (0)$. If $t(l) = 0$ for an $l \in L$, then it is easy to see that $l = xl_1$ for some $l_1 \in L$. Hence, $f(l) = xf(l_1) = 0$ and using that x is a X_C-regular element, $f(l_1) = 0$. Then we can continue this process and show that $l \in \bigcap_{i=1}^{\infty} x^i L$. Since L is a finitely generated R-module and $x \in \operatorname{rad} R$, we have that $\bigcup_{i=1}^{\infty} x^i L = (0)$ and therefore $\ker t = (0)$.

Next, we use the result above to characterize liftability of C in $\mathrm{CM}(\bar{R})$ in terms of existence of special submodules of X_C. But first we need to introduce some notation. Let $0 \to Y_C \to X_C \xrightarrow{\alpha_C} C \to 0$ be the minimal Cohen–Macaulay approximation of C in $\mathrm{CM}(\bar{R})$. Denote by $\pi: X_C \to \bar{X}_C$ the natural epimorphism and by $\bar{\alpha}_C: \bar{X}_C \to C$ the map $\bar{R} \otimes_R \alpha_C$ for $\alpha_C: X_C \to C$.

Proposition 4.2. *A module C in $\mathrm{CM}(\bar{R})$ is liftable if and only if there exists a submodule L of X_C, such that L is in $\mathrm{CM}(R)$, x is a X_C/L-regular element and the composition $\pi(L) \subsetneq \bar{X}_C \xrightarrow{\bar{\alpha}_C} C$ is an isomorphism.*

Proof. Assume that L in $\mathrm{CM}(R)$ is a lifting of C in $\mathrm{CM}(\bar{R})$. By Proposition 4.1 we have the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & L \arrow[r, equal] \arrow[d, "i"] & L \arrow[d] & \\
0 \arrow[r] & Y_C \arrow[r] & X_C \arrow[r, "\alpha_C"] & C \arrow[r] & 0,
\end{tikzcd}$$

and applying the functor $\bar{R} \otimes_R -$, the following commutative diagram is induced:

$$\begin{tikzcd}
0 \arrow[d] & & \\
\mathrm{Tor}_1^R(\bar{R}, X_C/L) \arrow[d] & & \\
\bar{L} \arrow[r, equal] \arrow[d, "\bar{i}"] & \bar{L} \arrow[d, "\wr"] & \\
\bar{X}_C \arrow[r, "\bar{\alpha}_C"] & C \arrow[r] & 0.
\end{tikzcd}$$

Since $\bar{\alpha}_C \circ \bar{i}$ is an isomorphism, $\mathrm{Tor}_1^R(\bar{R}, X_C/L) = (0)$ and therefore x is a X_C/L-regular element. Since

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d] & X_C \arrow[d, "\pi"] \\
0 \arrow[r] & L/xL \arrow[r, "\bar{i}"] \arrow[d] & \bar{X}_C \\
 & 0 &
\end{tikzcd}$$

is a commutative diagram, $\pi(L) = \pi(L) = \pi \circ t(L) = \bar{t}(L)$ and therefore the composition $\pi(L) \hookrightarrow \bar{X}_C \xrightarrow{\bar{\alpha}_C} C$ is an isomorphism.

The proof of the "if"-part of the statement follows from Lemma 2.3.

Now, we combine the characterization of liftable modules above with the characterization we found for liftable modules in Proposition 2.2 to get the following characterization of liftable Cohen–Macaulay modules over $\bar{R}$.

PROPOSITION 4.3. *Consider the following commutative diagram with C in* $\mathrm{CM}(\bar{R})$:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_R(C) \arrow[r] \arrow[d] & P \arrow[r, "\varphi"] \arrow[d, "t"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y_C \arrow[r] & X_C \arrow[r, "\alpha_C"] & C \arrow[r] & 0
\end{tikzcd}$$

The module C is liftable to R if and only if there exists a map $t\colon P \to X_C$ such that $\alpha_C \circ t = \varphi$, the composition $(\pi \circ t)(P) \hookrightarrow \bar{X}_C \xrightarrow{\bar{\alpha}_C} C$ is an isomorphism and x is a $X_C/\mathrm{Im}\ t$-regular element.

Proof. The "if"-part of the statement follows from Proposition 4.2.

Assume that L in $\mathrm{CM}(R)$ is a lifting of C to R. By Proposition 2.1 we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Omega_R(L) \arrow[r, equal] \arrow[d] & \Omega_R(L) \arrow[d] & & \\
0 \arrow[r] & \Omega_R(C) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "r"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega_R(L) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}.$$

By Proposition 4.1 we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L \arrow[r, "x"] \arrow[d] & L \arrow[r] \arrow[d, "s"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y_C \arrow[r] \arrow[d] & X_C \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & X_C/L \arrow[r, equal] \arrow[d] & X_C/L \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}.$$

If we let $t = s \circ r$, then t has the desired properties.

The preceeding result shows the close connection between an $\bar{R}$-module C in $\mathrm{CM}(\bar{R})$ and its minimal Cohen–Macaulay approximation X_C over R. Our next result shows that the category of liftable Cohen–Macaulay $\bar{R}$-modules is closed under syzygies, cosyzygies and taking dual. Every module C in $\mathrm{CM}(\bar{R})$ has a coresolution of the form

$$0 \longrightarrow C \longrightarrow \bar{R}^{n_0} \xrightarrow{d^1} \bar{R}^{n_1} \xrightarrow{d^2} \bar{R}^{n_2} \xrightarrow{d^3} \cdots,$$

where $\operatorname{Im} d^i$ is in $\mathrm{CM}(\bar{R})$ for all $i \geqslant 1$ and has no nonzero free summands. Then we call the module $\operatorname{Im} d^i$ an ith cosyzygy of C in $\mathrm{CM}(\bar{R})$ and denote it by $\Omega_{\bar{R}}^{-i}(C)$. The dual of a module C in $\mathrm{CM}(\bar{R})$ is defined by $D(C) = \operatorname{Hom}_{\bar{R}}(C, \bar{R})$.

PROPOSITION 4.4. *Let C be in $\mathrm{CM}(\bar{R})$. If C is liftable to R, then the syzygies $\Omega^i_{\bar{R}}(C)$ for any i in $\mathbf{Z}$ and the dual $\operatorname{Hom}_{\bar{R}}(C, \bar{R})$ of C are liftable to R.*

Proof. Let L in $CM(R)$ be a lifting of C in $\mathrm{CM}(\bar{R})$. Then $\overline{\Omega^i_R(L)} \simeq \Omega^i_{\bar{R}}(C)$ for all positive integers i, since tensoring a projective resolution of L over R with $\bar{R} \otimes_R -$ induces a projective resolution of $C \simeq \bar{R} \otimes_R L$. Therefore all the syzygies of C are liftable.

Applying $\operatorname{Hom}_R(L, \ )$ to the exact sequence $0 \to R \xrightarrow{x} R \to \bar{R} \to 0$ it induces the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(L, R) \xrightarrow{x} \operatorname{Hom}_R(L, R) \longrightarrow \operatorname{Hom}_{\bar{R}}(\bar{L}, \bar{R}) \longrightarrow 0,$$

since R is an injective module in $\mathrm{CM}(R)$. Hence, the dual of C given by

$D(C) \simeq D(\bar{L})$ is liftable, since the dual of L given by $D_R(L) = \mathrm{Hom}_R(L, R)$ is in $\mathrm{CM}(R)$. This also shows that taking dual and reduction modulo x commutes, that is, $\mathrm{Hom}_{\bar{R}}(\bar{L}, \bar{R}) \simeq \bar{R} \otimes_R \mathrm{Hom}_R(L, R)$.

Using the duality $\mathrm{Hom}_{\bar{R}}(\ , \bar{R}) : \mathrm{CM}(\bar{R}) \to \mathrm{CM}(\bar{R})$ we have that

$$\Omega_{\bar{R}}{}^{i}(C) \simeq \mathrm{Hom}_{\bar{R}}(\Omega^{i}_{\bar{R}}(\mathrm{Hom}_{\bar{R}}(C, \bar{R})), \bar{R}).$$

From this it follows immediately that the cosyzygies of C are liftable.

In the rest of this section we study the properties of the category of weakly liftable Cohen–Macaulay $\bar{R}$-modules which we denote by $\mathrm{w.l.}(\bar{R})$. Our aim is to show that $\mathrm{w.l.}(\bar{R})$ is a functorially finite subcategory in $\mathrm{mod}\,\bar{R}$. First we give a characterization of a weakly liftable Cohen–Macaulay $\bar{R}$-module C in terms of its minimal Cohen–Macaulay approximation X_C over R. Before we prove this we need to introduce some notation.

Let C be in $\mathrm{CM}(\bar{R})$. Then the minimal Cohen–Macaulay approximation of C over R can be constructed in the following way. Let $0 \to \Omega^1_R(D(C)) \to R^n \to D(C) \to 0$ be a projective cover of $D(C) = \mathrm{Hom}_{\bar{R}}(C, \bar{R})$ over R. It follows that $\Omega^1_R(D(C))$ is in $\mathrm{CM}(R)$ and that dualizing with respect to R induces the following exact sequence:

$$0 \to R^n \to \mathrm{Hom}_R(\Omega^1_R(D(C)), R) \to \mathrm{Ext}^1_R(D(C), R) \to 0. \ (*)$$

It is easy to check that $\mathrm{Ext}^1_R(D(C), R) \simeq \mathrm{Hom}_{\bar{R}}(D(C), \bar{R}) \simeq C$ and that the sequence $(*)$ is a minimal Cohen–Macaulay approximation of C over R. In the following we denote by $0 \to R^n \to X_C \xrightarrow{g} C \to 0$ the minimal Cohen–Macaulay approximation of C over R where $n = \mu(\mathrm{Ext}^1_R(C, R))$, the minimal number of generators of $D(C)$. Applying the functor $\bar{R} \otimes_R -$, we obtain an exact sequence

$$\xi_C : 0 \longrightarrow C \longrightarrow \bar{R}^n \longrightarrow \bar{X}_C \xrightarrow{\bar{g}} C \longrightarrow 0.$$

Then ξ_C represents an element in $\mathrm{Ext}^2_{\bar{R}}(C, C)$ and $\xi_C = \theta_C$, where

$$\theta_C : 0 \to C \to \Omega_R(C)/x\Omega_R(C) \to P/xP \to C \to 0$$

is induced from the exact sequence $0 \to \Omega_R(C) \to P \to C \to 0$ by tensoring with $\bar{R} \otimes_R -$ as defined in Section 1. Because, applying the functor $\bar{R} \otimes_R -$ to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega_R(C) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & R^n \arrow[r] & X_C \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd},$$

the following commutative diagram is induced:

$$\begin{tikzcd} \theta_C\colon 0 \arrow[r] & C \arrow[r] \arrow[d, equal] & \overline{\Omega_R(C)} \arrow[r] \arrow[d] & \bar{P} \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ \xi_C\colon 0 \arrow[r] & C \arrow[r] & \bar{R}^n \arrow[r] & \bar{X}_C \arrow[r, "\bar{g}"] & C \arrow[r] & 0 \end{tikzcd}.$$

This shows that θ_C and ξ_C represent the same element in $\mathrm{Ext}^2_{\bar{R}}(C, C)$.
As a consequence of Proposition 3.2 we have the following.

PROPOSITION 4.5. *The following are equivalent for C in* $\mathrm{CM}(\bar{R})$.

(a) C *is in* $\mathrm{w.l.}(\bar{R})$.

(b) $\xi_C = 0$ *in* $\mathrm{Ext}^2_{\bar{R}}(C, C)$.

(c) $\bar{g}$ *is a split epimorphism.*

(d) $\bar{X}_C \simeq C \oplus \Omega_{\bar{R}}^{-1}(C)$.

Proof. (a)$\Leftrightarrow$(b). Follows from Proposition 3.2 and above argument.

(a)$\Rightarrow$(c). Suppose C is in $\mathrm{w.l.}(\bar{R})$. Then there is an exact sequence $0 \to K \to L \xrightarrow{\varphi} C \to 0$ such that L is in $\mathrm{CM}(R)$ and φ induces a split epimorphism $\bar{L} \xrightarrow{\bar{\varphi}} C \to 0$. Then we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & L \arrow[r, "\varphi"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & R^n \arrow[r] & X_C \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd},$$

and applying the functor $\bar{R} \otimes_R -$ the following commutative diagram is induced:

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d, equal] & \bar{K} \arrow[r] \arrow[d] & \bar{L} \arrow[r, "\bar{\varphi}"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ \xi_C\colon 0 \arrow[r] & C \arrow[r] & \bar{R}^n \arrow[r] & \bar{X}_C \arrow[r, "\bar{g}"] & C \arrow[r] & 0 \end{tikzcd}.$$

Since $\bar{\varphi}$ is a split epimorphism, $\bar{g}$ is also a split epimorphism.

(c)$\Rightarrow$(d). Suppose $\bar{g}$ is a split epimorphism. We want to show that $\ker \bar{g} \simeq \Omega_{\bar{R}}^{-1}(C)$. The exact sequence $0 \to R \xrightarrow{x} R \to \bar{R} \to 0$ induces the following long exact sequence:

$$0 \to \mathrm{Hom}_R(C, R) \xrightarrow{x} \mathrm{Hom}_R(C, R) \to \mathrm{Hom}_R(C, \bar{R}) \to \mathrm{Ext}^1_R(C, R) \xrightarrow{x} \cdots .$$

Since $x \in \mathrm{Ann}_R(C)$, we have $\mathrm{Hom}_{\bar{R}}(C, \bar{R}) \simeq \mathrm{Ext}^1_R(C, R)$. Hence we have $n = \mu(\mathrm{Hom}_{\bar{R}}(C, \bar{R}))$ and $\ker \bar{g} \simeq \Omega_{\bar{R}}^{-1}(C)$.

(d) $\Rightarrow$ (a). Follows from the definition.

As consequences of this proposition, we haven the following corollaries which will be used later.

COROLLARY 4.6. *Let C be indecomposable in* $\mathrm{CM}(\bar{R})$ *and suppose that* $C \not\simeq \bar{R}$. *If C is weakly liftable to R, then X_C has no free summands.*

Proof. Since C is weakly liftable to R, we have $\bar{X}_C \simeq C \oplus \Omega_{\bar{R}}^{-1}(C)$ by (d) of Proposition 4.5. Since C is indecomposable in $\mathrm{CM}(\bar{R})$ and $C \not\simeq \bar{R}$, we know that $\Omega_{\bar{R}}^{-1}(C)$ is also indecomposable and $\Omega_{\bar{R}}^{-1}(C) \not\simeq \bar{R}$. So the module $\bar{X}_C$ has no free summands. Therefore X_C has no free summands. Since if F is a free summand of X_C then $\bar{F}$ is a free summand of $\bar{X}_C$.

COROLLARY 4.7. *Let C be in* $\mathrm{CM}(\bar{R})$ *and suppose C is indecomposable and not liftable. Assume* $\mathrm{CM}(\bar{R})$ *is a Krull-Schmidt category (for instance, when R is complete). If C is weakly liftable, then X_C is indecomposable.*

Proof. By (d) of Proposition 4.5 we have $\bar{X}_C \simeq C \oplus \Omega_{\bar{R}}^{-1}(C)$. Since C is indecomposable, $\Omega_{\bar{R}}^{-1}(C)$ is also indecomposable. Then if X_C is decomposable, we have that C is liftable which contradicts our assumption.

We end this section by showing that $\mathrm{w.l.}(\bar{R})$ is a functorially finite subcategory in $\mathrm{mod}\,\bar{R}$, a notion we now recall from [4]. Let $\mathscr{A}$ be a category and $\mathscr{C}$ a subcategory of $\mathscr{A}$. The subcategory $\mathscr{C}$ is said to be *contravariantly finite* in $\mathscr{A}$ if for any object A in $\mathscr{A}$, there is a C in $\mathscr{C}$ and a morphism $f: C \to A$ such that any morphism $g: C' \to A$ with C' in $\mathscr{C}$ factors through f. If the dual condition is satisfied for the subcategory $\mathscr{C}$, then $\mathscr{C}$ is said to be *covariantly* finite in $\mathscr{A}$. If $\mathscr{C}$ is both covariantly and contravariantly finite in $\mathscr{A}$, then $\mathscr{C}$ is said to be *functorially finite* in $\mathscr{A}$.

PROPOSITION 4.8. *The subcategory* $\mathrm{w.l.}(\bar{R})$ *is functorially finite in* $\mathrm{mod}\,\bar{R}$.

Proof. Given an $\bar{R}$-module C in $\mathrm{mod}\,\bar{R}$, we have an epimorphism $\bar{f}: \bar{X}_C \to C \to 0$. We want to show that for any D in $\mathrm{w.l.}(\bar{R})$ and any morphism $h: D \to C$, the morphism h factors through $\bar{f}$. Let L be in $\mathrm{CM}(R)$ such that there is a map $\varphi: L \to D$ which induces a split epimorphism $\bar{\varphi}: \bar{L} \to D$. Then the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & L & \xrightarrow{\varphi} & D & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & Y_C & \longrightarrow & X_C & \xrightarrow{f} & C & \longrightarrow & 0
\end{array}$$

induces the following commutative diagram by applying the functor $\bar{R}\otimes_R -$:

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r] \arrow[d] & \bar{K} \arrow[r] \arrow[d] & \bar{L} \arrow[r, "\bar{\varphi}"] \arrow[d] & D \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & C \arrow[r] & \bar{Y}_C \arrow[r] & \bar{X}_C \arrow[r, "\bar{f}"] & C \arrow[r] & 0 \end{tikzcd}.$$

Since $\bar{\varphi}$ is a split epimorphism, h factors through $\bar{f}$. This shows that w.l.$(\bar{R})$ is contravariantly finite in mod $\bar{R}$.

Now we want to show that w.l.$(\bar{R})$ is covariantly finite in mod $\bar{R}$. For C in mod $\bar{R}$ let C^* denote the $\bar{R}$-module $\mathrm{Hom}_{\bar{R}}(C, \bar{R})$. Then we have a natural morphism $\alpha_C: C \to C^{**}$ for every C in mod $\bar{R}$. Since w.l.$(\bar{R})$ is contravariantly finite in mod $\bar{R}$, we have a morphism $\beta: X \to C^*$ such that X is in w.l.$(\bar{R})$ and any morphism $h: D \to C^*$ with D in w.l.$(\bar{R})$ factors through β. Applying the functor $\mathrm{Hom}_{\bar{R}}(\;, \bar{R})$, we get $\beta^*: C^{**} \to X^*$. Let $g: C \to X^*$ be the composition $C \xrightarrow{\alpha_C} C^{**} \xrightarrow{\beta^*} X^*$. We claim that any morphism $h: C \to E$ with E in w.l.$(\bar{R})$ can be extended to X^*. Since E is reflexive, we have the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, "\alpha_C"] \arrow[d, "h"] & C^{**} \arrow[d, "h^{**}"] \\ E \arrow[r] & E^{**} \end{tikzcd}.$$

Since E^*, C^*, and X are reflexive $\bar{R}$-modules, applying the functor $\mathrm{Hom}_{\bar{R}}(\;, \bar{R})$ to the diagram

$$\begin{tikzcd} C^{**} \arrow[r, "\beta^*"] \arrow[d, "h^{**}"] & X^* \\ E^{**} & \end{tikzcd},$$

it induces the following commutative diagram:

$$\begin{tikzcd} & E^* \arrow[dl, "s"'] \arrow[d, "h^*"] \\ X \arrow[r, "\beta"] & C^* \end{tikzcd}.$$

Since w.l.$(\bar{R})$ is contravariantly finite in mod $\bar{R}$, the map h^* factors through β. Applying the functor $\operatorname{Hom}_{\bar{R}}(\ ,\bar{R})$ again, we have the commutative diagram

$$\begin{tikzcd} C^{**} \arrow[r, "\beta^*"] \arrow[d, "h^{**}"'] & X^* \arrow[dl, "s^*"] \\ E^{**} & \end{tikzcd}.$$

This shows $h: C \to E$ can be extended to X^* and w.l.$(\bar{R})$ is also covariantly finite in mod $\bar{R}$, and therefore w.l.$(\bar{R})$ is functorially finite in mod $\bar{R}$.

5. Weak Lifting over Gorenstein Rings

In this section, instead of studying the weak liftability of modules in $\mathrm{CM}(\bar{R})$, we study the weak liftability of modules in mod $\bar{R}$. A finitely generated $\bar{R}$-modules C is said to be weakly liftable to R if there exists an R-module L such that x is an L-regular element and C is isomorphic to a direct summand of L/xL. This section is mainly devoted to studying the properties of weakly liftable $\bar{R}$-modules in terms of their minimal Cohen–Macaulay approximations over R.

All the results in this section are based on the following lemma.

Lemma 5.1. *Let L be in* mod R *such that x is L-regular and let $0 \to Y_L \to X_L \longrightarrow L \to 0$ be the minimal Cohen–Macaulay approximation of L over R. Then the reduction $0 \to \bar{Y}_L \to \bar{X}_L \xrightarrow{\varphi} \bar{L} \to 0$ is the minimal Cohen–Macaulay approximation of $\bar{L}$ over $\bar{R}$.*

Proof. Since x is L-regular, the sequence $0 \to \bar{Y}_L \to \bar{X}_L \xrightarrow{\varphi} \bar{L} \to 0$ is exact. We know that x is regular on X_L and Y_L, so $\bar{X}_L$ is in $\mathrm{CM}(\bar{R})$ and $\bar{Y}_L$ is of finite projective dimension as an $\bar{R}$-module. Therefore $0 \to \bar{Y}_L \to \bar{X}_L \xrightarrow{\varphi} \bar{L} \to 0$ is a Cohen–Macaulay approximation of $\bar{L}$ over $\bar{R}$. It remains to show that it is minimal. This is done by using a general criterion of minimal Cohen–Macaulay approximation proved in [1]. It says that a Cohen–Macaulay approximation $0 \to Y \to X \xrightarrow{f} C \to 0$ of an R-module C is minimal if and only if X has a decomposition $F \oplus U$ with U no non-zero free summands and F a free module such that the induced map $F \to C/f(U)$ is a projective cover. Now suppose $X_L = F \oplus U$ is such a decomposition for X_L. Then we have $\bar{X}_L = \bar{F} \oplus \bar{U}$. It follows that $\bar{F}$ is a free

$\bar{R}$-module with rank $\bar{F} = \operatorname{rank} F$ and $\bar{U}$ has no free summands as an $\bar{R}$-module [1]. We now show that the induced map $\bar{F} - \bar{L}/\bar{\varphi}(\bar{U}) \to 0$ is a projective cover over $\bar{R}$. Since $\bar{L}/\bar{\varphi}(\bar{U}) \simeq L/(\varphi(U) + xL)$ and x is in the maximal ideal of R, we have that $\mu(\bar{L}/\bar{\varphi}(\bar{U})) = \mu(L/\varphi(U))$ and therefore $\bar{F} \to \bar{L}/\bar{\varphi}(\bar{U}) \to 0$ is a projective cover of $\bar{L}/\bar{\varphi}(\bar{U})$ over $\bar{R}$. By the criterion cited above we have showed that $0 \to \bar{Y}_L \to \bar{X}_L \xrightarrow{\varphi} \bar{L} \to 0$ is a minimal Cohen–Macaulay approximation of $\bar{L}$.

Let C be an arbitrary $\bar{R}$-module. We denote by $X_C^{\bar{R}}$ the minimal Cohen–Macaulay approximation of C over $\bar{R}$. As an immediate consequence of Lemma 5.1, we have the following result which shows the connection between the (weak) liftability of C and $X_C^{\bar{R}}$.

Proposition 5.2. *Let C be a finitely generated $\bar{R}$-module. If C is* (weakly) *liftable to R, then $X_C^{\bar{R}}$ and $Y_C^{\bar{R}}$ are* (weakly) *liftable to R.*

When $\dim R \leqslant 2$, the converse of Proposition 5.2 is also true. We state it in the following proposition.

Proposition 5.3. *Let* $\dim R \leqslant 2$ *and C be a finitely generated $\bar{R}$-module. Then C is (weakly) liftable to R if and only if $X_C^{\bar{R}}$ is (weakly) liftable to R.*

Proof. If $\dim R = 1$, then $\dim \bar{R} = 0$. So $X_C^{\bar{R}} = C$ and there is nothing to prove. Suppose $\dim R = 2$. Let $0 \to \bar{R}^n \to X_C^{\bar{R}} \to C \to 0$ be the minimal Cohen–Macaulay approximation of C over $\bar{R}$. Suppose $X_C^{\bar{R}}$ is liftable to R and suppose $0 \to X \xrightarrow{x} X \to X_C^{\bar{R}} \to 0$ is an exact sequence with X in $\mathrm{CM}(R)$. Then we have the following commutative exact diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & R^n \arrow[r, "x"] \arrow[d, "f"] & R^n \arrow[r] \arrow[d, "f"] & \bar{R}^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r, "x"] & X \arrow[r] & X_C^{\bar{R}} \arrow[r] \arrow[d] & 0. \\
 & & & C \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

By the Snake Lemma, we obtain an exact sequence

$$0 \to \operatorname{coker} f \xrightarrow{x} \operatorname{coker} f \to C \to 0.$$

Since x is regular on coker f, we have that C is liftable to R.

We have shown in Section 4 for C in $\mathrm{CM}(\bar{R})$, that if C is indecomposable and $C \not\simeq \bar{R}$, then C weakly liftable to R implies that X_C has no free summands. Now we generalize this result in the following form. Given an R-module C, we denote by $\delta_R(C)$ the maximal rank of free summands in X_C, the minimal Cohen–Macaulay approximation of C over R. We define $\delta^i_R(C) = \delta(\Omega^i_R(C))$. Since $\Omega^i_R(C)$ is in $\mathrm{CM}(R)$ and has no free summands for $i > \dim R$, we have $\delta^i_R(C) = 0$ for $i > \dim R$, so that the sum $\sum_{i \geq 0} (-1)^i \delta^i_R(C)$ makes sense.

PROPOSITION 5.4. *Let C be in* mod R. *If there is an R-regular element $x \in \mathrm{Ann}_R(C)$ such that, as an $\bar{R} = R/(x)$-module, C is weakly liftable to R, then $\sum_{i \geq 0}(-1)^i \delta^i_R(C) = 0$.*

Proof. Since C is weakly liftable to R, we have that $\overline{\Omega^1_R(C)} \simeq C \oplus \Omega^1_{\bar{R}}(C)$ by Proposition 3.2. Let $0 \to Y \to X \to \Omega^1_R(C) \to 0$ be the minimal Cohen–Macaulay approximation of $\Omega^1_R(C)$ over R. Then $0 \to \bar{Y} \to \bar{X} \to \overline{\Omega^1_R(C)} \to 0$ is the minimal Cohen–Macaulay approximation of $\overline{\Omega^1_R(C)} \simeq C \oplus \Omega^1_{\bar{R}}(C)$ over $\bar{R}$. Therefore we have $\bar{X} \simeq X^{\bar{R}}_C \oplus X^{\bar{R}}_{\Omega^1_{\bar{R}}(C)}$ by Lemma 5.1 and hence $\delta_{\bar{R}}(\overline{\Omega^1_R(C)}) = \delta_{\bar{R}}(C) + \delta_{\bar{R}}(\Omega^1_{\bar{R}}(C))$, in other words, we have the relation $\delta^1_R(C) = \delta^0_{\bar{R}}(C) + \delta^1_{\bar{R}}(C)$.

Let $\cdots \to P_1 \to P_0 \to \Omega^1_R(C) \to 0$ be an R-free resolution of $\Omega^1_R(C)$. Since x is a regular element on $\Omega^1_R(C)$, we have that $\cdots \to \bar{P}_1 \to \bar{P}_0 \to \overline{\Omega^1_R(C)} \to 0$ is an $\bar{R}$-free resolution of $\overline{\Omega^1_R(C)}$. Therefore we have $\overline{\Omega^{i+1}_R(C)} \simeq \Omega^1_{\bar{R}}(\overline{\Omega^1_R(C)})$ for all $i \geq 1$. Since C is weakly liftable to R as an $\bar{R}$-module, the syzygies $\Omega^i_{\bar{R}}(C)$ are also weakly liftable to R for all $i > 0$. Therefore by the above argument we have $\bar{X}_{\Omega^{i+1}_R(C)} \simeq X^{\bar{R}}_{\Omega^i_{\bar{R}}(C)} \oplus X^{\bar{R}}_{\Omega^{i+1}_{\bar{R}}(C)}$ for all $i \geq 0$ and hence $\delta^i_{\bar{R}}(C) + \delta^{i+1}_{\bar{R}}(C) = \delta^{i+1}_R(C)$ for all $i \geq 0$. Therefore the alternating sum of $\delta^{i+1}_R(C)$ for $i \geq 0$ is $\delta^0_{\bar{R}}(C)$. Hence we have $\sum_{i \geq 0}(-1)^i \delta^i_R(C) = \delta^0_R(C) - \delta^0_{\bar{R}}(C)$. Now we must show that $\delta_R(C) = \delta_{\bar{R}}(C)$. Since C is weakly liftable to R as an $\bar{R}$-module, $X^{\bar{R}}_C$, the minimal Cohen–Macaulay approximation of C over $\bar{R}$, is also weakly liftable to R by Proposition 5.2. Then our result follows from the following fact which we state as a lemma.

LEMMA 5.5. *Suppose C is an $\bar{R} = R/(x)$-module such that $X^{\bar{R}}_C$ is weakly liftable. Then $\delta_R(C) = \delta_{\bar{R}}(C)$.*

Proof. We write $X_C^{\bar{R}} = U \oplus \bar{R}^n$ where U is an $\bar{R}$-module without non-zero free summands. Let $X_U \to U$ be the minimal Cohen–Macaulay approximation of U over R. Then since U is in $\mathrm{CM}(\bar{R})$ and weakly liftable to R, we have $\delta_R(U) = 0$, that is, X_U has no free summands by Corollary 4.6. Now consider the following commutative exact diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & R^m \oplus R^n \arrow[r, equal] \arrow[d] & R^m \oplus R^n \arrow[d] & & \\
0 \arrow[r] & Z \arrow[r] \arrow[d] & X_U \oplus R^n \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0. \\
0 \arrow[r] & Y_C^{\bar{R}} \arrow[r] \arrow[d] & U \oplus \bar{R}^n \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\mathrm{pd}_{\bar{R}} Y_C^{\bar{R}} < \infty$, we have that $\mathrm{pd}_R Y_C^{\bar{R}} < \infty$. Therefore $\mathrm{pd}_R Z < \infty$ and $0 \to Z \to X \oplus R^n \to C \to 0$ is a Cohen-Macaulay approximation of C over R and we have $\delta_R(C) \leqslant n = \delta_{\bar{R}}(C)$. Since $\delta_{\bar{R}}(C) \leqslant \delta_R(C)$ for all $\bar{R}$-modules C, we obtain that $\delta_R(C) = \delta_{\bar{R}}(C)$ whenever $X_C^{\bar{R}}$ is weakly liftable to R.

Combining this proposition and Corollary 4.6, we have the following.

COROLLARY 5.6. *Let C be in $\mathrm{CM}(\bar{R})$ and $\bar{R}$ is not a summand of C. If C is weakly liftable to R, then $\delta^i_R(C) = 0$ for all $i \geqslant 0$.*

Let k denote the residue field $R/\underline{m}$. If k contains infinitely many elements, there exists an R-regular element $x \in \underline{m} \backslash \underline{m}^2$. As in Section 3, it follows that k as an $\bar{R} = R/(x)$-module is weakly liftable to R. Therefore we have $\sum_{i \geqslant 0} (-1)^i \delta^i_R(k) = 0$. In fact, more has been shown in [1]. We have the following.

PROPOSITION 5.7 [1]. *Let $k = R/\underline{m}$ be the residue field of R. Then $\delta^i_R(k) = 0$ for all $i \geqslant 0$.*

In view of these results we naturally ask whether the converse of Proposition 5.4 is true, that is, given an R-module C such that $\sum_{i \geqslant 0} (-1)^i \delta^i_R(C) = 0$ (or even $\delta^i_R(C) = 0$ for all $i \geqslant 0$) and such that $\mathrm{Ann}_R(C)$ contains R-regular elements. Then is there an R-regular element

$x \in \operatorname{Ann}_R(C)$ such that as an $\bar{R} = R/(x)$-module, C is weakly liftable to R. The answer for this question in general is negative as showed by the following example.

EXAMPLE. Let $R = \mathbf{C}[[t^3, t^4]]$ and $C = R/\underline{a}$, where $\underline{a} = (t^8, t^9)$. Then we know that $\delta^i_R(C) = 0$ for all $i \geqslant 0$ [7]. If there is an element $f \in \underline{a}$ such that C as an $\bar{R} = R/(f)$-module is weakly liftable to R, then we would have $\overline{(t^8, t^9)} \simeq C \oplus \Omega^1_{\bar{R}}(C)$. Now $\underline{a}$ has the presentation over R

$$R^2 \xrightarrow{\begin{pmatrix} t^8 & -t^9 \\ t^3 & -t^4 \end{pmatrix}} R^2 - \underline{a} \to 0,$$

and reducing this modulo f we obtain the following exact sequence:

$$\bar{R}^2 \xrightarrow{\begin{pmatrix} \bar{t}^8 & -\bar{t}^9 \\ \bar{t}^3 & -\bar{t}^4 \end{pmatrix}} \bar{R}^2 \to \bar{\underline{a}} \to 0.$$

There exist invertible matrices P and Q in $\operatorname{End}_{\bar{R}}(\bar{R}^2)$ such that

$$Q \begin{pmatrix} \bar{t}^8 & -\bar{t}^9 \\ \bar{t}^3 & -\bar{t}^4 \end{pmatrix} P = \begin{pmatrix} \bar{g} & 0 \\ 0 & \bar{z} \end{pmatrix}$$

with $\bar{g} \in \bar{\underline{a}}$. $(\bar{t}^8, \bar{t}^9, \bar{t}^3, \bar{t}^4) = \underline{\bar{m}} \neq (\bar{g}, \bar{z})$ as ideals in $\bar{R}$. This contradiction shows that $R/\underline{a}$ is not weakly liftable to R with respect to any element in $\underline{a}$.

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